Pricing Contingent Claims with an Underlying Asset Driven by an Extreme Value Distribution: Options on Dow Jones Industrial Average Index, 2009-2010

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Abstract

This paper is aimed to develop an equilibrium model in a stochastic economy populated by identical, competitive, and risk-averse consumers (investors), to value a European call option on stock whose price is subject to extreme and unexpected jumps. It is assumed that the underlying asset price is driven by a mixed jump-diffusion process with the jump size following an extreme value type distribution. The option price is characterized by a differential-integral equation with boundary conditions; an analytical solution for such a characterization is provided. Finally, a Monte Carlo simulation, with market daily data, is carried out to obtain numerical approximations of call options on the Dow Jones Industrial Average Index during the period 2009-2010.

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1- Introduction

There is a large literature on pricing options on jump-diffusions, see for instance: Cox and Ross (1976), Ball and Torous (1985), Page and Sanders (1986), Cao (2001), and more recently Chandrasekhar Reddy Gukhal (2004), among others. While this literature has provided considerable theoretical progress, it is missing a suitable analysis of the agents' rationality in pricing contingent claims when the underlying asset follows a mixed jump-diffusion process and the jump size is driven by an extreme value distribution.

One of the most important features of stock markets is that eventually there are sudden jumps, booms or crashes, and occasionally these sudden movements turn out to be of extreme magnitude. In fact, any investor is expecting a market boom of exceptional magnitude. Given the high volatility events recorded in 2009 in the financial markets in most of the world, and the consequences of the speculative bubble generated by those markets (including derivatives) in the real economy, it is essential to model the asset price dynamics using credible stochastic processes of extreme jumps and include them in scenarios where agents make portfolio decisions.

This investigation explicitly recognizes the occurrence of jumps of extreme size in the underlying asset in pricing a European call option, which provides significant results in the theory of log-optimal portfolios with derivatives. The option price is characterized by a differential-integral equation with boundary conditions, and an analytical solution for such a price is provided.

Kramkov and Schachermayer (1999) have shown the existence of log-optimal portfolios in a very general framework. Additionally, Goll and Kallsen (2000) and (2003) have obtained optimal solutions of maximizing expected logarithmic utility from consumption in terms of the semimartingale characteristics of the price process. On the other hand, Hurd, (2004) has extended the Merton's (1976) optimal portfolio selection problem within an exponential Levy market for an agent with logarithm utility giving explicit solutions. This paper extends Goll and Kallsen (2000), and Hurd (2004) work by providing the characterization of log-optimal portfolios under a mixed jump-diffusion process when the jump size is driven by an extreme value distribution.

Indifference pricing was introduced by Hodges and Neuberger (1989) and has received great attention recently. They paper is closely related to marginal utility-based pricing, sometimes known as Davis' (1998) fair pricing. Davis proposed a formula for the marginal utility-based price of a measurable European claim in incomplete markets. In exponential indifference pricing, it is well known that the marginal price of the claim is the expectation of the payoff (discounted) under the minimal entropy measure. When utility is logarithmic, the corresponding measure is the minimal martingale measure, and this measure is very helpful in quadratic hedging approaches. This paper is concerned in extending the option pricing method proposed by Davis (1998) for a competitive consumer with log-utility and provides an explicit construction of a fair price of a derivative in an incomplete market when the underlying asset price follows a Poisson process with its jump size governed by an extreme value distribution. Finally, a Monte Carlo simulation exercise, with actual DJIA index daily values, is carried out to obtain numerical approximations of call options on the Dow Jones Industrial Average Index (DJX) from January 1st 2009 to June 18th 2010.

The proposed model considers an economy populated by infinitely lived identical consumers that maximize utility from a single perishable good. Agents have access to three real assets: shares, European call options on such shares, and bonds paying a constant real interest rate, free of credit risk. It is assumed that underlying asset is driven by a mixed jump-diffusion process with the jump size guided by an extreme value distribution.[1]

The paper is organized as follows. In the next section, we work out an equilibrium model in a one-good, stochastic economy to value a European call option on stock whose price, in real terms, is driven by a geometric Brownian motion combined with a Poisson Process where the jump size is governed by an extreme value distributions. Through section 3, we solve the consumer-investor's decision problem. In section 4, we characterize the price of a European call option by a (partial) differential-integral equation with certain boundary conditions. In section 5, we derive an analytical solution for

[1] For the sake of simplicity, this research focus on upward jumps, that is, booms. The case of downward jumps can be easily extended by considering a second Poisson process with the jump size governed by an extreme value distribution of the Weibull type.

the derived differential-integral equation. In section 6, we provide numerical approximations of the option price using actual DJIA's index daily values from January 1st 2009 to June 18th 2010. Finally, in section 7, we present conclusions, acknowledge limitations, and make suggestions for further research.

2 - Setting of the economy

Let us consider an economy populated by infinitely lived identical consumer in a world with a single good. These agents make consumption and portfolio decisions.[2]

2.1 Dynamics of the underlying asset

Assume that agents have access to a share in real terms. Let us suppose that the price of such a share is driven by a geometric Brownian motion combined with Poisson where the jump size is governed by an extreme value distribution, that is

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \,\mathrm{d}W_t + \phi S_t \mathrm{d}N_t, \qquad (1)$$

where $\mu \in \mathrm{R}$, $\sigma > 0$, $(W_t)_{t \geq 0}$ is a Brownian motion defined on a fixed probability space $(\Omega, F, \mathrm{P}_W)$, and $\mathrm{d}N_t$ is a Poisson process with intensity parameter δ. It is assumed that processes $\mathrm{d}N_t$ and $\mathrm{d}W_t$ are uncorrelated, that is, $\mathrm{Cov}(\mathrm{d}N_t, \mathrm{d}W_t) = 0$.

Next, we outline, in detail, the characteristics of the jump component in equation (1). The size of an upward jump is defined as

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[2] Other papers dealing with an economy with variables driven by a geometric Brownian motion combined with a Poisson Process can be found in Venegas-Martínez (2001), (2006a), (2006b), (2008), (2009), (2010), and (2011), and Hernández-Lerma and Venegas-Martínez (2012). The present paper can be extended in several directions, for instance, by incorporating stochastic volatility as in Venegas-Martínez (2005) and/or by including and average consumer as in Venegas-Martínez (2013).

$$\phi = X^{-\alpha}, \qquad \alpha > 0, \qquad X = \frac{Y - \nu}{\kappa}, \qquad \kappa,\ \nu > 0, \tag{2}$$

where Y is a extreme value Fréchet random variable with parameters α , ν and κ . In this case, it is well known that the cumulative distribution function of Y is given by

$$F_Y(y) = \exp\left\{-\left(\frac{y-\nu}{\kappa}\right)^{-\alpha}\right\} \quad \text{if} \quad y < \nu, \tag{3a}$$

and $F_Y(y) = 0$ if $y \geq \nu$. In the case of a downward jump the Weibull distribution is used, and this is given by $F_Y(y) = 0$ if $y < \nu$, and

$$F_Y(y) = \exp\left\{-\left(\frac{(-y)-\nu}{\kappa}\right)^{-\alpha}\right\} \quad \text{if} \quad y \geq \nu. \tag{3b}$$

The density of the Fréchet distribution is:

$$f_Y(y) = \frac{\alpha}{\kappa} F_Y(y) \left(\frac{y-\nu}{\kappa}\right)^{-(1+\alpha)}, \qquad y \geq \nu, \tag{4}$$

The corresponding Weibull density distribution can be obtained in a similar way. In particular, if $\alpha > 2$, the Fréchet random variable satisfies

$$\mathrm{E}[Y] = \nu + \kappa\,\Gamma\left(1 - \frac{1}{\alpha}\right),$$

and

$$\mathrm{Var}[Y] = \kappa^2\left[\Gamma\left(1-\frac{2}{\alpha}\right) - \Gamma^2\left(1-\frac{1}{\alpha}\right)\right],$$

where $\Gamma(\cdot)$ is the Gamma function. On the other hand, the Poisson process $\mathrm{d}N_t$ with intensity δ satisfies

$$P_N\{\text{one unit jump during}\,\mathrm{d}t\} = P_N\{\mathrm{d}N_t = 1\} = \delta\mathrm{d}t$$

and

$$P_N\{\text{more than one unit jump during}\,\mathrm{d}t\} = P_N\{\mathrm{d}N_t > 1\} = o(\mathrm{d}t),$$

so that

$$P_N\{\text{no jump during}\,\mathrm{d}t\} = 1 - \delta\mathrm{d}t - o(\mathrm{d}t),$$

where $o(\mathrm{d}t)/\mathrm{d}t \to 0$ as $\mathrm{d}t \to 0$.

Notice now that the whole family of EV distributions can be expressed by using the gamma representation, due to Von Mises (1936). By defining the shape parameter $\gamma = 1/\alpha$, the cumulative distribution function and the density function can be, respectively, written as

$$F\left(y;\nu,\kappa,\gamma\right) = \exp\left\{-\left[1+\gamma\left(\frac{x-\nu}{\kappa}\right)\right]^{-\frac{1}{\gamma}}\right\},$$

and

$$f\left(y;\nu,\kappa,\gamma\right) = \frac{1}{\kappa}\left[1+\gamma\left(\frac{x-\nu}{\kappa}\right)\right]^{-\frac{1}{\gamma}-1}\exp\left\{-\left[1+\gamma\left(\frac{x-\nu}{\kappa}\right)\right]^{-\frac{1}{\gamma}}\right\}.$$

In this case, $\gamma > 0$ leads to the Fréchet distribution; while a $\gamma < 0$ leads to the Weibull distribution. The gamma representation is often used in statistical software (for more details see Reiss (2007)). It is important to point out that the parameter estimation made with the gamma representation can be easily converted into the original alpha parametrization used in this paper.

2.2 Available assets in the economy

The representative consumer holds three assets: a share priced in real terms as S_t, a European call option on S_t of value $V = V(S_t, t)$, and a real bond of price, b_t. The bond pays a constant real interest rate r (*i.e.*, it pays r units of the consumption good per unit of time). Thus, the consumer's real wealth, A_t, is given by

$$A_t = S_t + V(S_t, t) + b_t, \tag{5}$$

where A_0 is exogenously determined.

3 - Consumer's decision problem

Through this section, we state the decision intertemporal problem of a price-taking, risk-averse consumer-investor.

3.1 Consumer's budget constraint

Let us establish the consumer's budget constraint. Observe first that the stochastic rate of return of holding a share, $\mathrm{d}R_S = \mathrm{d}S_t / S_t$, is given by

$$\mathrm{d}R_S = \mu \mathrm{d}t + \sigma \mathrm{d}W_t + \phi \mathrm{d}N_t. \tag{6}$$

If $V = V(S_t, t)$ denotes the value of the option, then Itô 's lemma leads to

$$\mathrm{d}V = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_t} \mu S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma^2 S_t^2 \right) \mathrm{d}t + \frac{\partial V}{\partial S_t} \sigma S_t \mathrm{d}W_t + \left[V(S_t(\phi+1), t) - V(S_t, t) \right] \mathrm{d}N_t,$$

or

$$\mathrm{d}V = \mu_V V\mathrm{d}t + \sigma_V V\mathrm{d}W_t + \phi_V V\mathrm{d}N_t, \tag{7}$$

where

$$\mu_V = \frac{1}{V}\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}\sigma^2 S_t^2\right),$$

$$\sigma_V = \frac{1}{V}\frac{\partial V}{\partial S_t}\sigma S_t,$$

and

$$\phi_V = \frac{1}{V}\left[V(S_t(\phi+1),t) - V(S_t,t)\right].$$

In this case, the stochastic consumer's real wealth accumulation in terms of the portfolio shares, $w_{1t} = S_t / A_t, w_{2t} = V_t / A_t,\ 1 - w_{1t} - w_{2t} = b_t / A_t$, and consumption, c_t, becomes

$$\mathrm{d}A_t = A_t w_{1t}\mathrm{d}R_S + A_t w_{2t}\mathrm{d}R_V + A_t(1 - w_{1t} - w_{2t})r\,\mathrm{d}t - c_t\mathrm{d}t,$$

with A_0 exogenously determined. Notice now that the above budget constraint can be rewritten as

$$\begin{aligned}\mathrm{d}A_t = A_t\Bigg[& \left(r + (\mu - r)w_{1t} + (\mu_V - r)w_{2t} - \frac{c_t}{A_t}\right)\mathrm{d}t \\ & + (w_{1t}\sigma + w_{2t}\sigma_V)\mathrm{d}W_t + (w_{1t}\phi + w_{2t}\phi_V)\,\mathrm{d}N_t \Bigg].\end{aligned} \tag{8}$$

3.2 Consumer's satisfaction index

The von Neumann-Morgenstern utility at time $t=0$, v_0, of the competitive risk-averse consumer is assumed to have the time-separable form:

$$v_0 = \mathrm{E}_0\left[\int_0^\infty \log(c_t)e^{-rt}\mathrm{d}t\right], \tag{9}$$

where E_0 is the conditional expectation on all available information at $t=0$. To avoid unnecessary complex dynamics in consumption, we assume that the agent's subjective discount rate has been set equal to a constant interest rate, r. To obtain closed-form solutions and make the analysis simpler, we have considered the logarithmic utility function.

3.3 First-order conditions

The Hamilton-Jacobi-Bellman equation for the stochastic optimal control problem of maximizing utility, subject to constraint (8), is given by

$$\max_{w_{1t},w_{2t},c_t} H(w_{1t},w_{2t},c_t;A_t,t)$$

$$\equiv \max_{w_{1t},w_{2t},c_t}\Bigg\{ \log(c_t)e^{-rt} + I_A(A_t,t)A_t\left[r+(\mu-r)w_{1t}+(\mu_V-r)w_{2t}-\frac{c_t}{A_t}\right]$$

$$+I_t(A_t,t)+\frac{1}{2}I_{AA}(A_t,t)A_t^2(w_{1t}\sigma+w_{2t}\sigma_V)^2$$

$$+\delta\mathrm{E}_\phi\Big[I\big(A_t(w_{1t}\phi+w_{2t}\phi_V+1),t\big)-I(A_t,t)\Big]\Bigg\}=0, \tag{10}$$

where

$$I(A_t,t)=\mathrm{E}_t\left[\int_t^\infty \log(c_s)e^{-rs}\,\mathrm{d}\,s\right].$$

The first-order conditions for an interior solution are:

$$H_{c_t} = 0, \qquad H_{w_{1t}} = 0 \qquad \text{and} \qquad H_{w_{2t}} = 0.$$

We postulate indirect utility, $I(A_t, t)$, in a time-separable form as

$$I(A_t, t) = e^{-rt}[\beta_1 \log(A_t) + \beta_0],$$

where β_0 and β_1 are to be determined from (10). Hence, we obtain

$$\begin{aligned}\max_{w_{1t}, w_{2t}, c_t} H(w_{1t}, w_{2t}; A_t, t) \equiv \max_{w_{1t}, w_{2t}, c_t} \Big\{ \log(c_t) + \beta_1 \left[r + (\mu - r) w_{1t} + (\mu_V - r) w_{2t} - \frac{c_t}{A_t} \right] \\ -r[\beta_1 \log(A_t) + \beta_0] - \frac{1}{2}\beta_1 (w_{1t}\sigma + w_{2t}\sigma_V)^2 \\ +\delta\beta_1 \mathrm{E}_\phi \left[\log(w_{1t}\phi + w_{2t}\phi_V + 1) \right] = 0.\end{aligned}$$

If we now compute the first-order conditions for w_{1t} and w_{2t}, and interchange the order of the partial derivatives of w_{1t} and w_{2t} with the expectation operator, we find that

$$\mathrm{E}_\phi \left[\frac{\delta\phi}{w_{1t}\phi + w_{2t}\phi_V + 1} \right] + \mu - r = (w_{1t}\sigma + w_{2t}\sigma_V)\sigma$$

and

$$\mathrm{E}_\phi \left[\frac{\delta\phi_V}{w_{1t}\phi + w_{2t}\phi_V + 1} \right] + \mu_V - r = (w_{1t}\sigma + w_{2t}\sigma_V)\sigma_V.$$

Evidently, optimal consumption is proportional to real wealth at any time, that is, $c_t \propto A_t$.

4 - Characterization of the option price

In this section, we characterize the price of a European call option as the solution of a differential-integral equation. If we assume a corner solution, $w_{1t} = 1$ and $w_{2t} = 0$ (for pricing purposes), then the first-order conditions become, respectively,

$$\mu = r + \sigma^2 - \delta \mathrm{E}_\phi \left[\frac{\phi}{\phi + 1} \right], \tag{11}$$

and

$$\delta \mathrm{E}_\phi \left[\frac{\phi_V}{\phi + 1} \right] + \mu_V - r = \sigma \sigma_V. \tag{12}$$

From (12), it follows that

$$\delta \mathrm{E}_\phi \left[\frac{V(S_t(\phi+1),t) - V(S_t,t)}{\phi+1} \right] + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_t} \mu S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma^2 S_t^2 - rV = \frac{\partial V}{\partial S_t} \sigma^2 S_t.$$

If we use now (11) in the above expression, we get

$$\delta \mathrm{E}_\phi \left[\frac{V(S_t(\phi+1),t) - V(S_t,t) - \phi S_t \dfrac{\partial V}{\partial S_t}}{\phi+1} \right] + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_t} r S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma^2 S_t^2 - rV = 0. \tag{13}$$

We impose the boundary conditions $V(0,t) = 0$ and $V(S_t,t) = \max(S_t - K, 0)$ where K is the strike price of the option. In such a case, without loss of generality, we may consider a finite planning horizon $[0,T]$ in the expected utility expressed in (9). Notice that if $f_\phi(\cdot)$ is the density function of ϕ, the presence of the mathematical expectation in the above equation given by

$$\mathrm{E}_{\phi}\left[\frac{V(S_t(1+\phi),t)-\delta V(S_t,t)}{\phi+1}\right]=\int_{-\infty}^{\infty}\frac{V(S_t(1+\phi),t)-\delta V(S_t,t)}{\phi+1}f_{\phi}(\phi)\mathrm{d}\phi,$$

makes (13) a differential-integral equation. Observe also that if ϕ is constant in (13), by redefining δ as $\delta/(\phi+1)$, we obtain Merton's (1976) formula. Finally, observe that when $\phi=0$ or $\delta=0$, equation (13) reduces to the Black-Scholes' (1973) second order partial differential equation. Notice now that if we define the following change of variable:

$$u=\left(\frac{y-\nu}{\kappa}\right)^{-\alpha},$$

then one of the expectation terms in (13) satisfies

$$\mathrm{E}\left[\frac{\phi}{\phi+1}\right]=\mathrm{E}\left[\frac{X^{-\alpha}}{X^{-\alpha}+1}\right]$$

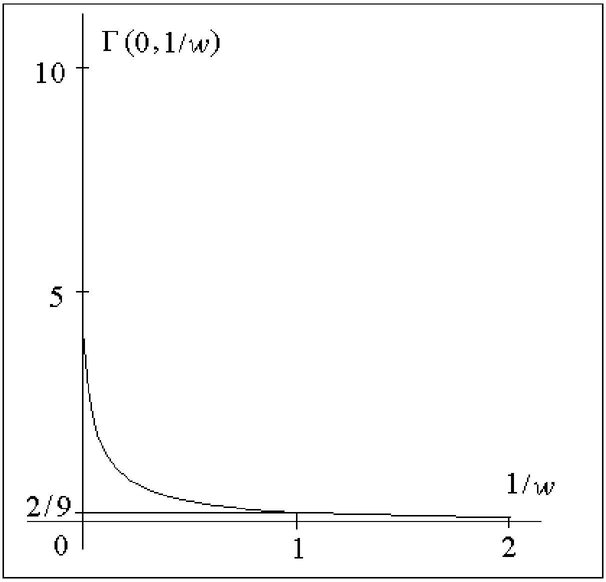
$$=\int_{0}^{\infty}\frac{u}{u+1}e^{-u}\mathrm{d}u$$

$$=e\,\Gamma(-1,1).$$

where $\Gamma(-1,1)=-\Gamma(0,1)+e^{-1}$, and $\Gamma(\cdot,\cdot)$ is the upper incomplete Gamma function. Figure 1 shows the graph of $\Gamma(0,1/w)$. It can be shown that $\Gamma(0,0)=\infty$, $\Gamma(0,\infty)=0$, and $\Gamma(0,1)\approx 2/9$ (in fact, $\Gamma(0,1)=0.219383934$). Therefore, equation (13) is transformed into

$$\delta\mathrm{E}_{\phi}\left[\frac{V(S_t(1+\phi),t)-V(S_t,t)}{\phi+1}\right]+\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S_t^2\frac{\partial^2 V}{\partial S_t^2}+S_t\frac{\partial V}{\partial S_t}\left[r+\delta\left(\frac{2}{9}e-1\right)\right]-rV=0.$$

Figure 1. Graph of the function $\Gamma(0,1/w)$.



Source: elaborated by the authors.

5 - Analytical solution of the option price

In order to determine the price $V = V(S_t, t)$, we define a sequence of random variables x_n, each with the distribution of the product of n independent and identically distributed random variables as $\phi + 1$, with $x_0 \equiv 1$. In other words, if $\{\phi_n\}_{n \in \mathrm{N}}$ is a sequence of independent and identically distributed random variables defined in (2), we set

$$x_n = \prod_{k=1}^{n} (\phi_k + 1).$$

In this case, the solution of equation (13) with the boundary conditions

$$V(0,t)=0, \qquad \text{and} \qquad V(S_t,0)=\max(S_t-K,0),$$

is given by

$$V(S_t,t)=\sum_{n=0}^{\infty}\mathrm{E}_{\phi}\mathrm{E}_{x_n}\left[\frac{e^{-\delta(T-t)/(\phi+1)}[\delta(T-t)/(\phi+1)]^n}{n!}V_{\mathrm{BS}}(S_t x_n e^{-\delta \mathrm{E}_{\phi}[\phi/(\phi+1)](T-t)},t;\sigma^2,r)\right], \tag{14}$$

where ϕ is independent of $\{\phi_n\}_{n\in\mathrm{N}}$ and $V_{\mathrm{BS}}(\cdot)$ is the basic Black-Scholes solution. Indeed, consider

$$V(S_t,t)=\sum_{n=0}^{\infty}\mathrm{E}_{\phi}\mathrm{E}_{x_n}[q_{n,t}V_{\mathrm{BS}}^{(n)}], \tag{15}$$

where

$$q_{n,t}=\frac{e^{-\delta(T-t)/(\phi+1)}[\delta(T-t)/(\phi+1)]^n}{n!},$$

$$U_{n,t}=x_n e^{-\delta \mathrm{E}_{\phi}[\phi/(\phi+1)](T-t)}$$

and

$$V_{\mathrm{BS}}^{(n)}=V_{\mathrm{BS}}(S_t U_{n,t},t).$$

In what follows, it is convenient to introduce the notation

$$Q_{n,t}=S_t U_{n,t}.$$

In such a case,

$$\frac{\partial V}{\partial S_t}=\sum_{n=0}^{\infty}\mathrm{E}_{\phi}\mathrm{E}_{x_n}\left[q_{n,t}U_{n,t}\frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial Q_{n,t}}\right], \tag{16}$$

$$\frac{\partial^2 V}{\partial S_t^2} = \sum_{n=0}^{\infty} \mathrm{E}_\phi \mathrm{E}_{x_n} \left[q_{n,t} U_{n,t}^2 \frac{\partial^2 V_{\mathrm{BS}}^{(n)}}{\partial Q_{n,t}^2} \right] \tag{17}$$

and

$$\begin{aligned} \frac{\partial V}{\partial t} = {} & \delta \mathrm{E}_\phi[\phi/(\phi+1)] \sum_{n=0}^{\infty} \mathrm{E}_\phi \mathrm{E}_{x_n} \left[q_{n,t} Q_{n,t} \frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial Q_{n,t}} \right] \\ & + \sum_{n=0}^{\infty} \mathrm{E}_\phi \mathrm{E}_{x_n} \left[q_{n,t} \frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial t} \right] \\ & + \delta \sum_{n=0}^{\infty} \mathrm{E}_\phi \mathrm{E}_{x_n} \left[\frac{q_{n,t} V_{\mathrm{BS}}^{(n)}}{\phi+1} \right] \\ & - \delta \sum_{n=1}^{\infty} \mathrm{E}_\phi \mathrm{E}_{x_n} \left[\frac{e^{-\delta(T-t)/(\phi+1)} [\delta(T-t)/(\phi+1)]^{n-1}}{(n-1)!} \left(\frac{V_{\mathrm{BS}}^{(n)}}{\phi+1} \right) \right] \end{aligned} \tag{18}$$

Hence, by virtue of (17) and (18), we get

$$\begin{aligned} \frac{\partial V}{\partial t} = {} & \delta \mathrm{E}_\phi[\phi/(\phi+1)] S_t \frac{\partial V}{\partial S_t} + \sum_{n=0}^{\infty} \mathrm{E}_\phi \mathrm{E} x_n \left[q_{n,t} \frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial t} \right] + \delta \mathrm{E}_\phi \left[\frac{V(S_t, t)}{\phi+1} \right] \\ & - \delta \sum_{m=0}^{\infty} \mathrm{E}_\phi \mathrm{E}_{Y_{m+1}} \left[\frac{e^{-\delta(T-t)/(\phi+1)} [\delta(T-t)/(\phi+1)]^{m}}{m!} \left(\frac{V_{\mathrm{BS}}^{(m+1)}}{\phi+1} \right) \right]. \end{aligned} \tag{19}$$

Observe that the last term in the above equation can be rewritten as

$$\begin{aligned} \mathrm{E}_\phi \left[\frac{V((1+\phi)S_t, t)}{\phi+1} \right] & = \sum_{n=0}^{\infty} \mathrm{E}_\phi \mathrm{E}_{x_n} \left[q_{n,t} \frac{V_{\mathrm{BS}}^{(n)}(Q_{n,t}(1+\phi), t)}{\phi+1} \right] \\ & = \sum_{n=0}^{\infty} \mathrm{E}_\phi \mathrm{E}_{Y_{n+1}} \left[q_{n,t} \frac{V_{\mathrm{BS}}^{(n+1)}(Q_{n+1,t}, t)}{\phi+1} \right] \end{aligned} \tag{20}$$

since $Q_{n+1,t}$ y $Q_{n,t}(1+\phi)$ are independent and identically distributed random variables. Therefore, equation (19) is transformed into

$$\frac{\partial V}{\partial t}=\sum_{n=0}^{\infty}\mathrm{E}_{\phi}\mathrm{E}_{x_n}\left[q_{n,t}\,\frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial t}\right]-\delta\mathrm{E}_{\phi}\left[\frac{V(S_t(\phi+1),t)-V(S_t,t)-\phi S_t\,\dfrac{\partial V}{\partial S_t}}{\phi+1}\right]. \quad (21)$$

From (16), (17) and (21), it follows that

$$\begin{aligned}&\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S_t^2\,\frac{\partial^2 V}{\partial S_t^2}+rS_t\,\frac{\partial V}{\partial S_t}-rV\\ &=\sum_{n=0}^{\infty}q_{n,t}\mathrm{E}_{x_n}\left[\frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial t}+\frac{1}{2}\sigma^2Q_{n,t}^2\,\frac{\partial^2V_{\mathrm{BS}}^{(n)}}{\partial Q_{n,t}^2}+rQ_{n,t}\,\frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial Q_{n,t}}-rV_{\mathrm{BS}}^{(n)}\right]\\ &\quad-\delta\mathrm{E}_{\phi}\left[\frac{V(S_t(\phi+1),t)-V(S_t,t)-\phi S_t\,\dfrac{\partial V}{\partial S_t}}{\phi+1}\right]. \qquad (22)\end{aligned}$$

Since

$$\frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial t}+\frac{1}{2}\sigma^2Q_{n,t}^2\,\frac{\partial^2V_{\mathrm{BS}}^{(n)}}{\partial Q_{n,t}^2}+rQ_{n,t}\,\frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial Q_{n,t}}-rV_{\mathrm{BS}}^{(n)}=0$$

holds for all $n\in\mathrm{N}\cup\{0\}$, we deduce, immediately, that (14) is solution of (13). Finally, in order to derive the put-call parity, we emphasize in the notation of (15) that we are dealing with a call option by writing

$$V_{\mathrm{call}}=\sum_{n=0}^{\infty}\mathrm{E}_{\phi}\mathrm{E}_{x_n}\left[q_{n,t}\,V_{\mathrm{BS,call}}^{(n)}\right] \qquad (23)$$

and under the same assumptions, it can be shown that a put option satisfies

$$V_{\text{put}} = \sum_{n=0}^{\infty} \mathrm{E}_{\phi} \mathrm{E}_{x_n} \left[q_{n,t} V_{\text{BS,put}}^{(n)} \right]. \tag{24}$$

Then, the put-call parity is obtained by inserting in the expectations of equation (23) the identity

$$V_{\text{BS,call}}^{(n)} = V_{\text{BS,put}}^{(n)} + S_t U_{n,t} - K e^{-r(T-t)}. \tag{25}$$

In the next section put option prices will be computed by using (25).

6 - Numerical approximations

In order to obtain numerical approximations of (14), the parameters of the extreme value distributions are estimated by using a maximum likelihood procedure available in the Xtremes3.01 software, and the quantity inside the mathematical expectations in (14)

$$M_{\phi,Y_n} = \sum_{n=0}^{10{,}000} \frac{e^{-\delta(T-t)/(\phi+1)}[\delta(T-t)/(\phi+1)]^n}{n!} V_{\text{BS}}^{(n)}$$

is simulated by using the statistical software “R” (R project for statistical computing, GNU Project) and Ripley's methodology (1987) for Monte Carlo simulations.[3] Subsequently, we compute the average of 10,000 simulated values of M_{ϕ,Y_n} to obtain, for different values of δ, approximate solutions of the option price. To do this, we first use the parameter values in Table 1 for computing the classical Black-Scholes price $V_{\text{BS}}^{(0)}$ for different values of the Dow Jones Industrial Average (DJIA) index; we set a dollar value to each *DJIA* index point. Table 2 shows numerical approximations of the option price by using the Merton's jump-diffusion formula for different DJIA values. Finally, Table 3 provides the results of a Monte Carlo simulation for different values of δ with $\mathrm{E}_{\phi}[\phi/(\phi+1)] = -e\Gamma(-1,1)$. It is assumed, for simulation

[3] The pricing algorithm code in R Statistical Software is available to interested readers.

purposes, that ϕ follows an extreme value distribution whose mean and variance are given by $-\ 0.05600967$ and $-\ 0.06181319$, respectively, all of the other values are kept equal for comparison purposes.

Table 1. Parameter values and prices for the benchmark Black-Scholes option price.

Source: elaborated by the authors.

K		r		σ		$T-t$	
10450		0.0018		0.0154		0.25	
Different values for the *DJIA*, S_t							
10450	10460	10470	10480	10490	10500	10510	10520

Table 2. Parameter values and prices for the benchmark Merton's option price.

Source: elaborated by the authors.

	K	r	σ	$T-t$	λ	max iter.		
	10450	0.002	0.015	0.25	0.0093	20		
	Different values for the *DJIA*, S_t							
K	10450	10460	10470	10480	10490	10500	10510	10520
1	44.67	56.49	69.99	50.36	63.03	44.67	56.49	69.99
2	49.18	60.6	73.55	54.69	66.89	49.18	60.6	73.55
3	53.38	64.54	77.1	58.78	70.65	53.38	64.54	77.1
4	57.27	68.26	80.55	62.6	74.24	57.27	68.26	80.55
5	60.89	71.77	83.86	66.18	77.66	60.89	71.77	83.86
6	50.36	63.03	44.67	56.49	69.99	50.36	63.03	44.67
7	54.69	66.89	49.18	60.6	73.55	54.69	66.89	49.18
8	58.78	70.65	53.38	64.54	77.1	58.78	70.65	53.38
9	62.6	74.24	57.27	68.26	80.55	62.6	74.24	57.27

Table 3. Parameter values and prices of call options with an underlying asset following a jump size from an extreme value distribution.

Source: elaborated by the authors.

	K	r	σ	$T-t$	λ	Iter.	ν	κ	α
	10450	0.002	0.016	0.25	0.009	20	-0.056	-0.061	-4.32
	Different values for the *DJIA*, S_t								
K	10450	10460	10470	10480	10490	10500	10510	10520	10530
1	42.57	48.17	54.2	60.66	67.54	74.82	82.46	90.45	98.74
2	45.18	50.5	56.22	62.33	68.84	75.73	82.99	90.58	98.5
3	47.6	52.72	58.21	64.07	70.28	76.86	83.78	91.04	98.62
4	49.81	54.79	60.1	65.76	71.75	78.07	84.72	91.69	98.98
5	51.81	56.68	61.86	67.35	73.16	79.28	85.71	92.45	99.48
6	53.64	58.41	63.48	68.85	74.51	80.46	86.7	93.24	100.1
7	55.31	60.01	64.99	70.24	75.77	81.58	87.67	94.03	100.6
8	56.85	61.49	66.38	71.53	76.95	82.64	88.59	94.81	101.2
9	58.28	62.85	67.67	72.74	78.07	83.64	89.47	95.56	101.8
10	59.61	64.13	68.88	73.87	79.11	84.59	90.31	96.27	102.5

It is important to point out that the option prices in Table 3 depend of the choice of the mean and the variance of the random variable ϕ. In this case, they were obtained from a maximum likelihood estimation process and by using market values. We may conclude, from Table 3 and the chosen mean and variance, that the option price increases when the average number of jumps per unit of time increases since a growing δ rises the future opportunity cost of the underlying asset. Finally, put option prices are computed by using the put-call parity as expressed in equation (23)-(24). To do this, we use different values of the *DJIA* index and set a dollar value to each *DJIA* index point.[4]

[4] The estimation routine is available to interested readers.

Table 4. Put option prices by using the put-call parity with an underlying asset following a jump size from an extreme value distribution.
Source: elaborated by the authors.

DJX	10450	10460	10470	10480	10490	10500	10510	10520	10530
Call	42.57	48.17	54.2	60.66	67.54	74.82	82.46	90.45	98.74
Put	43.84	48.93	61.28	61.13	69.21	74.93	85.31	96.27	102.5

7 - Conclusions

By extending the research in Davis (1998), Goll and Kallsen (2000), and Hurd (2004), we have developed an equilibrium stochastic model, with a representative and competitive risk-averse consumer, useful for pricing contingent claims on jump-diffusions when the jump size driven by a extreme value type distribution. The option price was characterized by a differential-integral equation with boundary conditions and an analytical solution was provided. Finally, a Monte Carlo simulation was carried out to obtain numerical approximations of call options on the Dow Jones Industrial Average Index (DJX) during the period 2009-2010. Needless to say, extensions considering jumps in the volatility parameter instead of jumps in the underlying price should be considered in future research.

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