Money Supply in a Simple Economic Growth Model and Multiple Steady States Equilibria

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Abstract

We intend to examine a monetary economic growth model à la Sidrauski-Brock where money is introduced both in the utility function and the production function. We assume that utility is derived from the flow of services derived from real money holdings and that money is held by firms to facilitate production. We show that the level of equilibrium depends on the rate of discount. We give conditions to guarantee uniqueness of the equilibrium. We demonstrate that the unique equilibrium is either a classical « saddle point » or a « source point »; that the introduction of money in the utility and production function is sufficient to produce multiple stationary equilibria; and that the initial stock of money appears as an important economic control variable. Therefore, the initial amount of the monetary emission issued by the central bank becomes essential for the long run economic equilibrium properties.

Keywords: economic growth, money, multiple equilibria.
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1 - Introduction

This study provides some insight about the conditions and the consequences related to the introduction of money in the neoclassical model of economic growth. This analysis focuses on the properties of the long term monetary equilibrium with the inclusion of money both on the utility and the production function. This question has been addressed notably by Solow and Orphanides (1990) as well as McCallum and Goodfriend (1987) in a survey about the role of money in the theory of economic growth. This makes it possible to tackle the complementary relationship between consumption goods and money, developed by Blanchard and Fisher (1989)[1]. For Solow and Orphanides as well as Blanchard and Fisher, the analysis is developed in the framework of the canonical model initiated by Sidrauski in 1967. Prior to build up our version of the Sidrauski model, we first present a short survey of the problems related to the inclusion of money in the neoclassical model of economic growth. The extensions of the Sidrauski model with regard to the other methods of monetary analysis are defined. The introduction of money in a barter economy is not always optimal in spite of the services provided by this asset in the exchange. This paradox is due to a large extent to the fact that money is a particular asset since people don't get from it a direct utility in consumption. The long term equilibrium with money *à la* Solow-Swan (1956) is not Pareto optimal in comparison with the real equilibrium. In this kind of model, in the long run and everything being equal, the inclusion of an external money reduces the level of capital per head [e.g. Levahri and Patinkin (1968), Sidrauski (1967a)]. *" The mere existence of the outside money, independent of the rate at which the money stock grows or declines through time, will prevent the economy from attaining the Solow-Swan steady state capital stock. The capital stock for which a monetary economy reaches its steady state is always smaller than the one indicated by the Solow-Swan model."* p.796, Sidrauski (1967a). Indeed, holding liquidities gives rise to a reduction in the saving devoted to the physical capital accumulation. This is the traditional capital-money eviction effect. Nevertheless, numerous authors consider that this situation is somewhat paradoxal since in the long run, money as a means of exchange has positive effects on economic growth. *"(...) equilibrium intensity is lower in the monetary model (...) this conclusion seems unreasonable. For if the sole result of introducing money into an economy were to reduce k and hence per capita output and consumption, why should it be introduced ?"* p. 717, Levahri and Patinkin (1969). Consumers get indeed

[1] Chapter 4.

a certain utility from the services related to money because this asset makes it possible to buy consumer goods. In addition, holding liquidity is also due to the precautious motivation. The contribution of money can be shown by a functional relationship between consumption and the amount of liquidity held by consumers. In most cases, this relationship is taken into account through the definition of a general utility function in which real money balance is taken as an argument. In particular, this method is presented in the pioneer article of Sidrauski (1967), extended by Brock (1974) and Calvo (1979). The innovation in the Sidrauski-Brock model (S-B) is based on explicit microeconomic behaviours. Although this method seems today reasonable, it was not the case in the past. As a matter of fact, money was generally introduced in terms of partial equilibrium with aggregated variables (for instance, refer to Tobin (1965)). Monetary theory was also mainly characterised by institutional studies [e.g. Friedman, (1969)][2]. While inspired by the Ramey model (1928) about the optimal saving behaviour, Sidrauski introduces a break with these methods through the application of the dynamic optimization technique for describing money demand. Like Cass (1965) and Koopmans (1965), the author considers an enlarged family with an infinite life duration (immortal family) which maximises its utility function with an intertemporal income constraint. Since then, the S-B model has proved to be particularly adapted for studying the existence, the uniqueness and the stability of the long run equilibrium with the inclusion of money assuming perfect expectations. Although more recent developments do not contradict these properties, the monetary models with cash in advance *à la* Lucas-Stockey (1987) complement the knowledge of equilibria with money[3]. The abundant literature related to the S-B model can be classified into two kinds of studies: some focus on the question of the existence, the uniqueness and the stability of the long run monetary equilibrium in a model of economic growth [e.g. Calvo (1979), Fisher (1979), Black (1974), Brock (1974)]. The others deal with money neutrality in the sense defined by Friedman [e.g. Woodford (1990), Petrucci (1999), Friedman (1969)]. For example, these authors are interested in the effects of inflation on economic growth or in the question of the optimal amount of money. This latter group of researches contributed for a large part to the current literature in monetary economics. However, the

[2] It is worthwhile mentioning that the monetary properties related to the generation model proposed by Allais (1947) and above all Samuelson (1958) have not been often taken into account in the monetary debates in the 60s.

[3] In this regard, let's cite Woodfort (1990): *" Because the Lucas-Stokey model is formally isomorphic to a particular case of the Sidrauski-Brock model (...) ",* p. 1121, op.cit.

properties related to the existence, the uniqueness and the stability of the monetary equilibrium are not contested any longer[4]. Looking at this literature indicates however that the monetary equilibrium properties in a model of economic growth can be further investigated from a theoretical and technical point of view. As already mentioned at the beginning of the introduction, this is the aim of the present article. Looking at various surveys, such as Solow and Orphanides (1990) as well as the reference articles, shows that the S-B model is still incomplete since money is introduced either in the utility function or in the production function. In other words, to our knowledge, no study analyses a monetary economy where money is simultaneously introduced in the utility and the production function. This topic also makes possible to tackle the complementary relationship between money and consumer goods in the utility function. As it will be shown later, this still open question [e.g. Blanchard and Fisher (1989), Brock (1974)] has certain consequences on the properties of the monetary equilibrium. Finally, although we find many justifications concerning the introduction of money in the utility function, there are also certain arguments for the introduction of this asset in the production function. As a matter of fact, Lehvari and Patinkin (1969) suggest the following analysis: *" (...) by this we mean that money is held because it enables the economic unit in question to acquire or produce a larger quantity of commodities, in the usual sense of the term.(...) Thus real money balance can be considered just like any other inventory which enters into the productive process."*, p.737, Lehvari and Patinkin (1969).

In their surveys, some authors like Orphanides and Solow (1990) admit that technical problems in modelling explain to a large extent the absence of such a study. We propose to develop this study in a S-B type model, in which money is included both in the utility and in the production function. We present a property concerning the existence of the monetary equilibrium, as a complement of standard results developed by Calvo (1979), Fisher (1979), Brock (1974). Our results highlight the conditions for the existence of a monetary equilibrium as well as the role of the complementary relationship between money and consumer goods. The local study about the steady-state shows that the equilibrium is a saddle point like most of the monetary models with two dimensions [e.g. Burmeister and Dobell (1970), ch.6]. There exists at least one steady-state equilibrium as the subjective

[4] Hahn (1965) is one of the first authors who tackle the question of the existence and uniqueness of the monetary equilibrium. In a framework which is different for the S-B model, this author is searching for the conditions which ensure the existence of a general economic equilibrium in which the price of money is positive.

discount rate for the representative agent is high enough. In other words, the excessive capital accumulation due to a too small preference rate for the present leads to a positive growth rate in the long run. However, this property is not sufficient to ensure the equilibrium uniqueness. Indeed, we show that the equilibrium uniqueness with money is not always ensured. Moreover, the existence of multiple monetary equilibria can lead to threshold effects. Such effects due to multiple equilibria have given rise to an increasing interest of the literature concerning economic growth theory which takes into account human capital externalities [e.g. Azariadis and Drazen, (1990)]. In particular, the authors outline the relationship between multiple equilibria and a particular threshold effect: the underdevelopment trap. In the present study, we find threshold effects without the underdevelopment trap mechanism *à la* Azariadis and Drazen. This is due to the particular nature of the steady-states. The threshold effect obtained in our study is nevertheless important because the initial amount of available money determines the long run economic equilibrium properties. The first section presents the central assumptions concerning the inclusion of money in the utility and the production function. In particular, we develop the microeconomic foundations of the agents' money demand. The second section focuses on the calculation of the intertemporal monetary equilibrium in a model *à la* Sidrauski-Brock. We finally exhibit the properties related to the existence and the uniqueness of the equilibrium with money. We show in this section the possibility of multiple steady states equilibria.

2 - The introduction of money in the function of utility and the production function

We present in this section the introduction of money in the production function and utility function. With regard to integration in the utility function, we can count at least two integration methods, direct and indirect. In this study, we will retain direct integration because it is the most commonly used. But Howitt (1993) uses the S-B model where money is taken into account through an explicit modelling of transactions.[5] Similarly, we suppose in this contribution that money is directly integrated into the production function as a factor of production. Moreover, in this model money is government debt. It is a claim held by consumers and firms against government. From the standpoint

[5] A point in that spirit is made by Howitt (1993) even if the author presupposes the existence of a unique equilibrium.

of the private sectors, it is a net external or outside claim[6]. Money earns no interest[7]. We shall also assume that there are no direct storage costs of holding money. Similarly, as long as we shall deal with outside money, we shall assume that there are no costs of administering the monetary system. Resolution of the use and holding of money is not easy. We can interpret money balances either as a consumer's good or as a producer's good. In the first case, the services rendered by money balances should appear in the individual's utility; whereas in the second, from producer's viewpoint, money holdings per se are assumed not to generate any utility. We assume that a producer without money would have to devote effort in order to achieve the multitude of "double coincidences" [8] [e.g. Lehvari and Patinkin (1968)]. The production function has the following form [e.g. Orphanides and Solow (1990)] $Y = F(K, L, M)$, where *Y* is the output, *K* the capital, *L* labor force, and M the real money balance. The production function is homogeneous of degree 1 in capital, labor and money and has the usual neoclassical properties. Per capita output *Y/L* can therefore be expressed as a function of the capital-labor ratio *k*. $y = f(k, m)$. The production function f is twice continuously differentiable, positive, increasing and strictly concave. The Inada conditions are satisfied. We assume that the marginal product of money is positive and diminishing; i.e., $f_m(k,m) > 0$, $f_{mm}(k,m) < 0$, and $f_{kk}(k,m) f_{mm}(k,m) - f_{km}(k,m)^2 > 0$. [9] For simplicity, there is no physical depreciation of capital. We assume that money is held only because it enables the producers to acquire or produce a larger quantity of commodities. Real money balances are found to be a factor input that cannot be treated as separable from the primary factor inputs capital and labor [e.g. Patinkin (1956)]. Furthermore, we assume that real money balance is complementary with capital $f_{km}(k,m) > 0$ [e.g. Siandra (1990)]. The concavity of production function, and $f_{km}(k,m) > 0$ involve the two following inequalities $f_{kk}(k,m) - f_{km}(k,m) < 0$, and $f_{mm}(k,m) - f_{km}(k,m) < 0$. If Orphanides and Solow (1990) incorporate the real money balance in the production function following analysis of Patinkin, we can note an important

[6] See Grandmont (1985) for the definitions of inside and outside money.
[7] See Fisher (1972a).
[8] See Diamond (1984) for a modern evaluation of the double-coincidence in a monetary-search economy.
[9] See Mas-Colell *et alii* (1995).

difference about the nature of that asset. According Patinkin, money can not be incorporated in both the production and the utility functions. The integration in the production function assumes that money is an intermediary-input whose services cannot be equated with those of a final product. Thus, money has a "derived" or "indirect" productivity, emphasizing the fact that money switches real resources from the exchange activity to the production activity. The treatment of the balance as a factor of production implies that the demand for money is determined by the marginal productivity principle. For simplicity, Patinkin and Lehvari (1967) assume that money is directly introduced into the production function.

Let us now briefly consider the alternative of treating money as a consumer's good in the utility function. Note that the utility function, u, evaluates the per capita consumption, c, and the real money balance m. Let $u(c,m): \mathrm{R}_+^2 \to \mathrm{R}$ be utility function. By assumption, this function is concave and twice differentiable for $c \in \mathrm{R}_+$, and $m \in \mathrm{R}_+$; thus $u_m(c,m) > 0$, $u_{mm}(c,m) < 0$, and $u_{cc}(c,m)u_{mm}(c,m) - u_{cm}(c,m)^2 > 0.$ [10] For any m, the utility function satisfies the following conditions: $\lim u_c(c,m) \to 0$ where $c \to +\infty$, and $\lim u_c(c,m) \to +\infty$ where $c \to 0$. In contrast, the properties of the utility function are more specific for real money balance. For any c, we have the following properties: $\lim u_m(c,m) \to 0$ where $m \to +\infty$, and $\lim u_m(c,m) < +\infty$ where $m \to 0$. This last hypothesis means that the marginal utility of money is limited for $c > 0$. In other words, the substitution between consumption and money is limited. We assume that money and consumption are « complements » $u_{cm}(c,m) > 0$ [11]. We are studying the possibility of complementarity in this study, because Brock (1974) has already demonstrated the multiplicity of equilibria with stationary currency in the case where consumption and money are perfectly substitutable ($u_{cm}(c,m) < 0$ [12]).

[10] We let to the reader the task to of verifying this property linked to the Hessian matrix.

[11] As we have seen in introduction, the consequences of this hypothesis are not known [e.g. Blanchard and Fisher (1989), chapter 4].

[12] Brock (1974) indicates that the multiplicity of equilibria may be proved in the case of complementarity between money and consumption goods without offering a demonstration. Whatever the assumption on the link between money and consumption goods, the author doesn't consider money as the productive factor when making his study.

To sum up, the method which consists of including money both in the production and the utility functions is important for the remainder of this study. The monetary asset is included following two approaches. From the consumer standpoint, liquidity provides an indirect service since its makes the transaction easier for economic agents. On the individual's side, the overall money services are valued within the utility function. At the same time, money directly contributes to production because it can be considered as an input in the production function. As a result, it seems reasonable to distinguish two components, namely the monetary asset for the consumer as well as the quantity of money devoted to production. The introduction of *M/L* into the utility function is anticipated earlier by the discussion of the dual role of money. It could be conceivably argued that money should be broken into two parts: the fraction of money that provides direct production services and the fraction that provides consumption services. If quantities are proportional, then the presentation is adequate. If this proportionality does not hold, then the allocation between production function and utility function could be considered to be part of the optimization problem. This procedure is certainly possible but lacks realism: money seems to produce "joint" services. This joint service is reflected in the appearance of the same *M* in the production and utility function.

3 - Monetary equilibrium in the Sidrauski-Brock model

Let us consider the S-B model with money in the production function. We assume that labor grows at a constant rate *n*, so that if L_0 is the labor force at time *t=0*, then $L(t) = e^{n.t} L_0, \quad L_0 > 0$. The economy is populated by infinitely lived families. Families have perfect foresight. The household can hold its wealth in the form of either money or physical capita. The consumption, *c*, and real money balance, *m*, per capita are control variables.[13]. The preferences of the representative family for consumption and real money balance over time are represented by the utility integral:

$$W_0 = \int_0^\infty e^{-\rho s} u(c(t), m(t)) ds \,,$$

subject to the constraints :

[13] We can remark that the number of states variables and the number of controls need not be equal [e.g. Kamien and Schwartz (1981)].

$$\dot{a}(t) = f(k(t), m(t)) + x - n\,a(t) - c(t),$$
$$a(t) = k(t) + m(t), \quad \text{for all } t,$$

where x is the real value of transfer payments to the family, and a the family wealth *per capita*. The government transfer payments are assumed to be independent of the household's holdings of money-so that they do not affect his decisions with respect to the level of these holdings. The parameter ρ is the rate of time preference, which is assumed to be strictly positive. The first equation is the wealth accumulation equation. The change in family wealth ($da(t)/dt$) is equal to output and transfer payments minus spending (on consumption and the amount of wealth). The second constraint is the family wealth, which is equal to holdings capital plus family real money balance per capita. We assume that the family cannot borrow and lend. The objective of the representative household is to maximize the welfare function. We assume that the central planner wants at time *t=0* to maximize family welfare. We characterize the solution of the central planner's program using the Maximum Principle [e.g. Pontryagin *et alii* (1962)]**.** We use more particularly the proposition of Arrow and Kurz (1970) modified by Skiba (1978).

Proposition 1. Let $\{c^*(t), m^*(t)\}$ ***be a choice of decision (t*** ≥ 0 ***) which maximizes***

$$\int_0^{\infty} e^{-\rho t} u(c(t), m(t))\,dt$$

subject to the conditions

(i-a) $\dot{a}(t) = f(k(t), m(t)) + x - n\;a(t) - c(t),$

and constraint,

(i-b) $(a(t) - m(t) - k(t)) \geq 0$

involving the instruments and possibly the state variables, initial conditions on the state variables, and the non-negativity condition

(ii) $a(t) \geq 0$**, with** $t \geq 0$**,**

on the state variable. If the constraint qualification holds, then there exist functions of time $\theta(t)$ ***such that, for each t,***

(iii) $\{c^*(t), m^*(t)\}$ ***maximizes*** $\mathcal{H}\,(a(t), c, m, \theta(t))$ ***subject to the constraint*** **(1-b),** ***where***

$$\mathcal{H}(m,\ c,\ a\ ,\theta) = u(m,\ c) + \theta\ (f(k,m) + x - na - c)\ ;$$

(iv) $\dot{\theta} = \theta(\rho + n) - \frac{\partial \mathrm{L}}{\partial a}$, ***evaluated at*** $a = a(t)$, $c = c^*(t)$, $m = m^*(t)$, $\theta = \theta(t)$, $\lambda = \lambda(t)$, ***and***

(v) $\mathbf{L}(m, c, k, \theta, \lambda, t) = \mathcal{H}(m, c, k, \theta, t) + \lambda(a - m - k)$.

and the Lagrangian multiplier λ ***is such that,***

(vi) $\frac{\partial \mathrm{L}}{\partial c} = \frac{\partial \mathrm{L}}{\partial m} = 0$, ***for*** $a = a(t)$, $c = c^*(t)$, $m = m^*(t)$, $\theta = \theta(t)$, $\lambda(t) \geq 0$, $\lambda(t)\left(a(t) - m^*(t) - k(t)\right) = 0.$

(vii) $\lim_{t \to +\infty} \theta(t)\, a(t) = 0$, ***(transversality condition).***

We use the necessary conditions of proposition 1 to solve a control problem involving two state variables, two control variables, and an equality constraint on the control variables. In this model, consumption and real money balance cannot be greater than wealth. So we introduce a constraint on the values that the control variables *c* and *m* may take on at any point of time. We shall consider the equivalence between centralized and decentralized economies.

Proposition 2. There exists equivalence between the decentralized and decentralized short term equilibrium.

As a first step, we calculate the centralized monetary equilibrium over a short horizon. Write the current-value Lagrangian:

$$L(m, c, k, \theta, \lambda) = u(c, m) + \theta\left(f(k, m) + x - n\,a - c\right) + \lambda(a - m - k),$$

And maximize with respect to *c*, *m*, *k*, and λ. The first-order conditions are:

$$\frac{\partial\, \mathrm{L}(m,\ c,\ k, \theta, \lambda)}{\partial c} = 0, \ \Leftrightarrow u_c(c, m) - \theta = 0, \tag{1a}$$

$$\frac{\partial\, \mathrm{L}(m,\ c,\ k, \theta, \lambda)}{\partial m} = 0, \ \Leftrightarrow u_m(c, m) + \theta\, f_m(k, m) - \lambda = 0, \tag{1b}$$

$$\frac{\partial\, \mathrm{L}(m, c, k, \theta, \lambda)}{\partial k} = 0, \Leftrightarrow \theta\ f_k(k, m) - \lambda = 0, \tag{1c}$$

$$\frac{\partial\, \mathrm{L}(m,\ c,\ k, \theta, \lambda)}{\partial \lambda} = 0, \ \Leftrightarrow a - k - m = 0. \tag{1d}$$

Summing the relations (1b) and (1c), and eliminating λ gives:

$$u_m(c,m)+\theta\ \left(f_m(k,m)-f_k(k,m)\right)=0\,.$$

This yields:

$$u_c(c,m)-\theta\ =0\,,$$

$$u_m(c,m)+\theta\ \left(f_m(k,m)-f_k(k,m)\right)=0,$$

$$a-k-m=0\,.$$

This is a system of three equations in three unknowns: *c*, *k*, and *m*. The properties of *u* and *f* ensure the existence and uniqueness of an interior solution. We shall write the solution for this problem in the form $c\left(\theta,a\right)$, $k\left(\theta,a\right)$, $m\left(\theta,a\right)$ for a given shadow price θ and wealth *a*. We must show that the dynamic behaviour of the decentralized economy will be the same as that of centrally planned. Now for any given *a*, the following maximization program has a solution: $\max\limits_{k,m} f\left(k,m\right)$,

subject to: $a\ =k+m$. The first order conditions yield the system A:

$$\begin{aligned} &f_k(k,m)-f_m(k,m)=0,\\ &a=k+m \end{aligned} \qquad \text{(A)}$$

This is a system of two equations in two unknowns. We shall write the solution for this problem in the form: $k=k^*(a)$, and $m=m^*(a)$. These two demand equations yield the optimal allocation of the existing wealth between real money balance and physical capital. From the strict concavity of *f*, it follows that these maximizing values are positive finite for any positive *a*. We can choose an arbitrary wealth a_0. We shall show that the existence of a short term equilibrium with $k=k^*(a_0)$, and $m=m^*\left(a_0\right)$. To show this, note the first short-term equilibrium is defined by the following system B:

$$u_c\left(c,m\right)=\theta\,,$$

$$f_m(k,m)-f_k\left(k,m\right)=\frac{u_m(c,m)}{u_c(c,m)}\,, \qquad \text{(B)}$$

$$a=k+m.$$

We evaluate this system for the fixed values $k=k^*(a_0)$, $m=m^*\left(a_0\right)$, and a_0. Now it is necessary to set up the model as a decentralized competitive economy. As we let $c\ \to 0$, we have $\theta\ \to +\infty$ or $u_c\left(c,m\right)\to +\infty$. By assumption, the marginal utility of money is bounded ($u_m\left(c,m\right)<+\infty$); thus $\dfrac{u_m\left(c,m\right)}{u_c\left(c,m\right)}\rightarrow 0$, we have

$$a_0=k^*\left(a_0\right)+m^*\left(a_0\right),$$
$$f_k\left(k^*\left(a_0\right),m^*\left(a_0\right)\right)-f_m\left(k^*\left(a_0\right),m^*\left(a_0\right)\right)=0.$$

This system is consistent with (A).The analysis of comparative statics of the short-term equilibrium will clarify some important points pertaining to the dynamics analysis (C.f. Appendix). The system (B) represents the unique short-term equilibrium. Thus, for any time *t*, *k* and *p* are given, then the system (B) yields essentially the derived demand equations for each moment of time as functions of θ and *a*. We can write the system solved: $c\left(\theta,a\right)$, $k\left(\theta,a\right)$, $m\left(\theta,a\right)$. It is important to note that the control variables *m* and *k* do not have continuous functions; indeed the optimal policy calls in general for a discontinuous transfer of money (i.e. a jump) at time 0 to correct the initial allocation of money. In other words, the allocation of a given wealth a between capita land money is governed by equation system (B). Since a and θ are continuous functions of time, *k* and *m* will also continuous for *t*>0; but the historically given *m(0)* and *k(0)* need not satisfy (B), so that, a jump may take place then [e.g. Sargent and Wallace (1973), Arrow and Kurz (1971)].

Proposition 3. The marginal utility of money is positive when the marginal productivity of capital is higher than that of money $f_k\left(k,m\right)>f_m\left(k,m\right)$**.**

The demonstration is simple. We use the first order conditions obtained in the previous proposition, so we can write

$$u_m\left(c,m\right)=-u_c\left(c,m\right)\ \left(f_m\left(k,m\right)-f_k\left(k,m\right)\right),$$
$$f_k\left(k,m\right)-f_m\left(k,m\right)=\frac{u_m\left(c,m\right)}{u_c\left(c,m\right)}>0\ \Rightarrow f_k\left(k,m\right)-f_m\left(k,m\right)>0.$$

Interpreting that marginal productivity of capital is higher than that of money needs to be discussed. Several cases can be examined. First, if the production elasticities with respect to capital and with respect to money are assumed to be equal, then the previous condition implies that the quantity of money held by the agent must be relatively higher than the physical capital. For a given level of savings, the agent has to choose between a relatively high quantity of money with low return and quantity of capital with high marginal productivity return. If savings increases, then the resulting consumption decrease is mainly due to the supplementary money demand. Therefore the services offered by the amount of money compensates for the decrease in consumption. This property can be interpreted in the light of the monetary theory of Brunner and Meltzer (1971). These authors show that real money balance improves information to increase quantity of consumption goods. In accordance with the principles of welfare economics, the services of real money balance must clearly be considered with respect to the alternative cost. There is indeed an optimal arbitrage between money and the consumer goods: *"The marginal cost of information is the consumption sacrificed",* p. 795. One can assume in a more realistic manner, that the elasticity of output relative to capital is greater than that of money. In this case the previous condition is verified for relatively large quantities of money and physical capital. This seems more realistic because it characterizes better complementarity between these two assets. This property has been confirmed by empirical testing. By the early seventies, Sinai and Stokes (1972), (1975), (1981), (1989) began to estimate a Cobb-Douglas production function containing real annual data based on data from developed economies. The major motivation of these researches concerning real money balances as a factor of production was to attempt to capture the effects of developments in the financial sector on real output. Their results were then amended and confirmed in the case of a developing economy by Khan and Ahmad (1985). These contributions show that capital and the real money balance have a comparable impact. The introduction of money reduces the relative weight attached to labour in the production function. More recently, the importance of money in the production function has been established within the framework of a model of endogenous growth by Evans, Green and Murinde (2002).

4 - Multiple steady states equilibria and threshold effect in a monetary economy

The long term equilibrium path is characterized by the differential equation of the auxiliary variable (i.e shadow price) and the differential equation of the physical capital. In order to examine the nature of the possible stationary points, consider the following system (**S**):

$$\dot{\theta} = \phi(\theta,a) = \theta(\rho+n) - \theta f_k\left(k(\theta,a),m(\theta,a)\right),$$

$$\dot{a} = \psi(\theta, a) = f\left(k(\theta, a), m(\theta, a)\right) + x - na - c(\theta, a).$$

From the Pontryagin principle we obtain the first differential equation which governs the behavior of θ. The second equation is the capital accumulation equation. We shall start by studying the system dynamical S. The functions ϕ and ψ are of class C^2 for $\theta \in R_+$, and $a \in R_+$. All the partial derivatives of the derived demand equations $c(\theta, a)$, $k(\theta, a)$, and $m(\theta, a)$ are analyzed in the Appendix.

Definition*. *The long-term monetary equilibrium is characterized by the two following conditions: Let two sets $E_1 = \{(\theta, a) \in R_+^2 \mid \dot{\theta} = \phi(\theta, a) = 0\}$***, and*** $E_2 = \{(\theta, a) \in R_+^2 \mid \dot{a} = \psi(\theta, a) = 0\}$ ***;The long-term equilibrium set is defined by the following relation*** $E_1 \cap E_2 \neq \varnothing$ ***for*** $(\theta, a) \in R_+^2$***.*** From this definition, comparative dynamics can be made. Let us suppose that the discount rate ρ is a changeable characteristic of the model.

Proposition 4. If discount rate ρ is "sufficiently large", then there exist at least one local stationary point. The proof can be established by using the implicit function theorem. By assumption, the continuous function $\phi(\theta, a)$ is defined on an open set *U*. The partial derivative $\frac{\partial \phi(\theta, a)}{\partial \theta}$ is continuous on *U*, and there exists a point $(\hat{\theta}, \hat{a}) \in U$, such that $\phi(\hat{\theta}, \hat{a}) = 0$, and $\frac{\partial \phi(\hat{\theta}, \hat{a})}{\partial \theta} \neq 0$. Thus, there exists two open sets U_1, and U_2, with $\theta \in U_1$, and $a \in U_2$ such that the dynamics of shadow price at the steady state $\phi(z(a), a) = 0$, with $z(a) \in U_1$, allows us to find a function *z* on U_2 to U_1. By the continuity of *z*, we get $z(\hat{a}) = \hat{\theta}$. The function $\phi(\theta, a)$ is of class C^1 on *U*, *z* is of class C^1 on U_2. $\phi(\theta, a)$ is differentiable at *a* on U, and *z* is differentiable at *a*. Hence

$\frac{d\theta}{da} = -\left(\frac{\partial \phi(z(a), a)}{\partial \theta}\right)^{-1} \left(\frac{\partial \phi(z(a), a)}{\partial a}\right)$. Now, we must find the sign of $\frac{d\theta}{da}$ by studying the right-hand side of the preceding equation

$$\frac{\partial \phi\left(z(a),a\right)}{\partial \theta} = -\left(f_{kk}\left(k,m\right)k_\theta + f_{km}\left(k,m\right)m_\theta\right).$$

We obtain $k_\theta = -m_\theta$ (Appendix):

$$\frac{\partial \phi\left(z(a),a\right)}{\partial \theta} = m_\theta\left(f_{kk}\left(k,m\right) - f_{km}\left(k,m\right)\right).$$

By assumption, we must have $m_\theta < 0$, $f_{kk}(k,m) < 0$, and $f_{km}(k,m) > 0$, thus proving that

$$\frac{\partial \phi\left(z(a),a\right)}{\partial \theta} > 0 \tag{2}$$

To justify the sign of $\frac{\partial \phi\left(z(a),a\right)}{\partial a}$, we use the following relation:

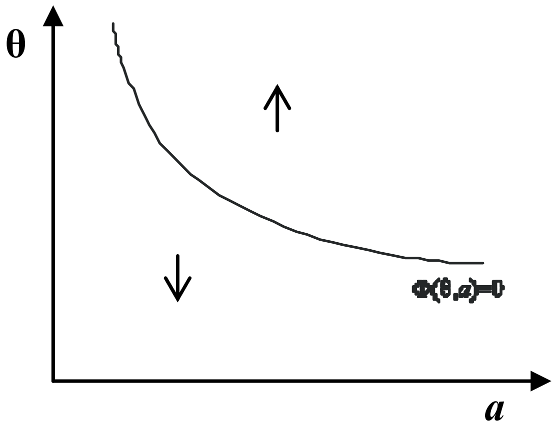
$$\frac{\partial \phi\left(z(a),a\right)}{\partial a} = -\left(f_{kk}\left(k,m\right)k_a + f_{km}\left(k,m\right)m_a\right),$$

where $k_a = 1 - m_a$ ($m_a = \partial m / \partial a$). After some algebraic manipulations, we obtain the sign of the following relation:

$$\frac{\partial \phi\left(z(a),a\right)}{\partial a} > 0 \tag{3}$$

Thus, by using expressions (2), and (3), we have demonstrated that $\frac{d\theta}{da} < 0$. This means that there exists an inverse relation between the shadow price and the stock of wealth. The function $\phi\left(z(a),a\right) = 0$ can be described in Figure 1:

Figure 1



We now consider the function $\psi(z(a),a)=0$ (Appendix):

$$\frac{\partial \psi(z(a),a)}{\partial \theta} = k_\theta(f_k - f_m) - c_\theta > 0$$

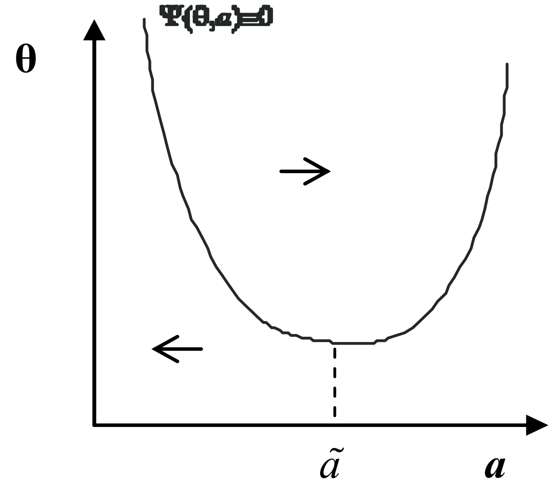
$$\frac{\partial \psi(z(a),a)}{\partial a} = f_m m_a + f_k k_a - c_a - n = f_k + (f_m - f_k)m_a - c_a - n.$$ The last two equations allow us to derive the following relation: $\frac{d\theta}{da} = -\left(\frac{\partial \psi(z(a),a)}{\partial \theta}\right)^{-1}\left(\frac{\partial \psi(z(a),a)}{\partial a}\right)$. The function $\psi(z(a),a) = 0$ is convex on (0, +∞). Since $(f_m - f_k) < 0$, $m_a > 0$, and $k_a > 0$ it follows that there exists a finite stock of wealth $\tilde{a}$ (i.e. a critical point), such that $\frac{d\theta}{da} = 0 \Leftrightarrow \frac{\partial \psi(z(\tilde{a}),\tilde{a})}{\partial a} = f_k + (f_m - f_k)m_a - c_a - n = 0, \quad \forall n \in \left]0,1\right[.$

Let O an open set in the neighbourhood of the point $\tilde{a}$, there exists a stock of wealth $\underline{a} \in O$ with $\underline{a} < \tilde{a}$, such that

$\dfrac{\partial \psi(\theta,\underline{a})}{\partial a} = f_k + (f_m - f_k)m_a - c_a - n > 0$, and the stock of wealth $\overline{a} \in O$ with $\overline{a} > \widetilde{a}$, such that $\dfrac{\partial \psi(\theta,\overline{a})}{\partial a} = f_k + (f_m - f_k)m_a - c_a - n < 0$.

The function $\psi(\theta,a) = 0$ is convex in the wealth *a* in the plane (θ,a):

Figure 2



We now consider the relations between $\phi(\theta,a) = 0$, and $\psi(\theta,a) = 0$. The curve $\phi(\theta,a) = 0$ lies above the curve $\psi(\theta,a) = 0$ for a given a. Without loss of generality, we can fix the stock of wealth on some arbitrary level $a = \widetilde{a}$. Given the assumptions about the production function, the local inversion theorem allows us to write θ_1 as a function of $\widetilde{a}$, and $(\rho + n)$ with $\phi(\theta,a) \in E_1$. This yields:

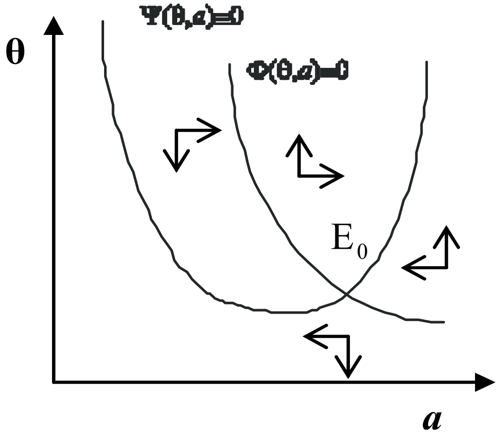
$$(\rho + n) = f_k(\theta_1;\widetilde{a}),$$
$$\theta_1 = f_k^{-1}(\rho + n;\widetilde{a}).$$

We now consider the function $\psi(\theta,a)$. Given the wealth level, $\widetilde{a}$, there exists θ_2 such that $f(\theta_2,\tilde{a}) + x - n\tilde{a} - c(\theta_2,\tilde{a}) = 0$, for $\theta_2 \in [0,\infty[$. Thus, there exists a value of the discount rate, $\overline{\rho}$, "sufficiently large", such that the inequalities are verified for θ_1 , and θ_2 $\psi(\theta_2,\tilde{a}) < \phi(\theta_1,\tilde{a}) \Leftrightarrow \theta_2 = g(\tilde{a}) < \theta_1 = f(\tilde{a}\ ;\bar{\rho})$.

The change in the population growth rate has a similar effect to that of the discount rate. With regard to these two rates, it is worth mentioning that they are not determined by the government. Only a demographic policy can provide an inflexion to the population growth rate. However, the long run population dynamic is very difficult to control. Concerning the discount rate, it is difficult to control as well even if the government can slightly influence it, through a planning policy. In spite of this limit, let's go on with this analysis starting from the discount rate. Amongst all the possible economic solutions, we can focus on the long run monetary equilibriums through a graphical analysis[14]. From this analysis, it can be shown how the kind of long term monetary equilibrium depends both on the preference parameter ρ and the functions ϕ, and ψ slopes. In particular, we observe that if the functions are monotonic and decreasing on $]0,\tilde{a}[$, then for a given value of ρ, there is at least one long run monetary equilibrium. From a geometric point of view, the kind of equilibrium depends on the ratio of the functions ϕ and, ψ slopes. In Figure 3, the unique stationary equilibrium E_0 is a saddle-point steady state. In this case, $\left.\frac{d\phi}{da}\right|_{a_0} > \left.\frac{d\psi}{da}\right|_{a_0}$, $\forall a_0 \in]0,\tilde{a}[$ (cf. Figure 3).

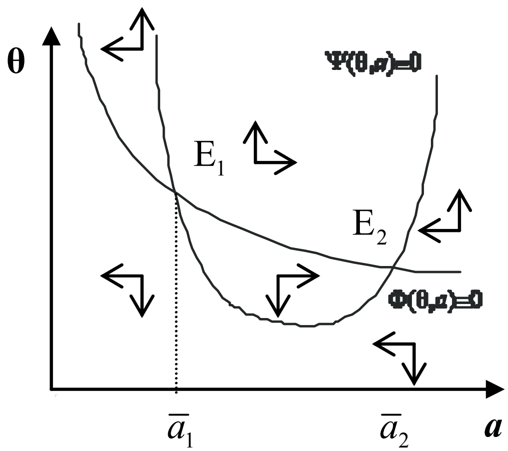
[14] In many cases an exact closed-form solution of the system **S** can't be found [e.g. Fisher (1978)]. Yet the solution can be characterized qualitatively by diagrammatic analysis Kamien and Schwartz (1981), Léonard and Van Long (1992).

Figure 3: **A unique equilibrium with a saddle point E_0.**



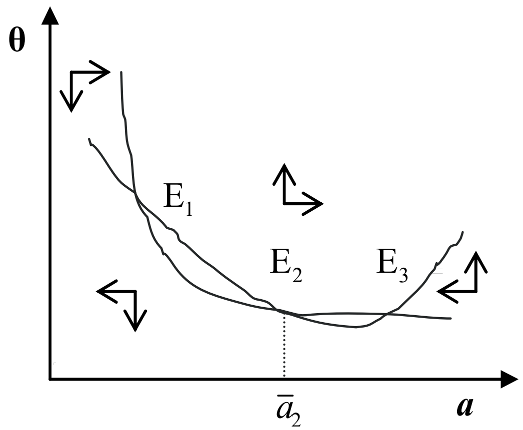
There exits at least one local equilibrium if the slope of ψ is steeper than the slope of ϕ; $\left.\frac{d\phi}{da}\right|_{a_0} < \left.\frac{d\psi}{da}\right|_{a_0}$, $\forall a_0 \in \left]0,\tilde{a}\right[$. In Figure 4, there exist multiple stationary equilibria. The number of equilibria depends on the relationship between the slope of ϕ $\left.(d\phi/da)\right|_{a_0}$, and the slope of ψ $\left.(d\psi/da)\right|_{a_0}$. Thus, if $\left.(d\phi/da)\right|_{a_0} < \left.(d\psi/da)\right|_{a_0}$ there exists an even number of stationary points. Two points, E_1, and E_2, exist at which the curves $\phi(\theta,a)=0$ and $\psi(\theta,a)=0$ intersect. The interesting fact, explained by the arrows in Figure 4, is that the first point is a local unstable point, while the second is a saddle-point steady state.

Figure 4: **Multiple equilibria with E_1 a source point and E_2 a saddle-point.**



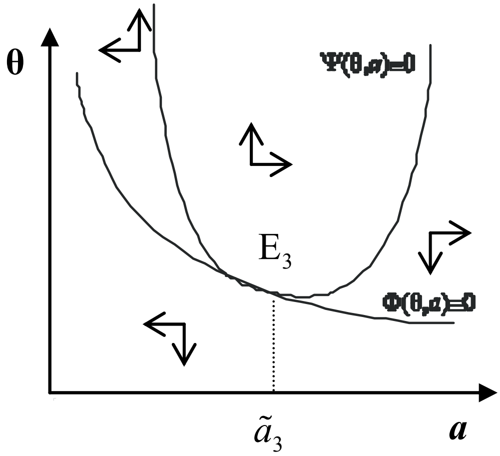
The steady state, E_1, a is high price equilibrium with small real money balance, and E_2 is a high price equilibrium large real money balance. At what equilibrium will the economy operate? It is clear that E_2 is a better equilibrium than E_1. In other words if $a(0)=\overline{a}_2$, the economy starts with a sufficient stock of wealth and converges to E_2. In Figure 5, the stability-instability properties are modified, and there exists an odd number of stationary points ($\left.(d\phi/da)\right|_{a_0} > \left.(d\psi/da)\right|_{a_0}$). Figure 5 depicts the case of multiple equilibria with three stationary points.

Figure 5: **Multiple equilibria with E_1 and E_2 two saddle points and E_3 source point.**



Thus, the introduction of money in the function of production does not imply necessarily the existence of multiple equilibria. Multiple equilibria property for this type of model is pretty old since, as early as the 1960's, Kurz's contribution (1968) on the effects of wealth mentions the link between complementarity of the utility function arguments and the existence of multiple equilibria. But the introduction of money in the utility function and production function increases all the possible cases. Many configurations are a priori possible. There may exist one or more steady states. Some stationary points are saddle-points, while others are totally unstable points. Furthermore there exists another case where the equilibrium is a degenerate point. This situation is illustrated in Figure 6.

Figure 6**: Degenerate stationary point** E_3 **.**



When the two equilibria. E_1, and E_2 meet, they both vanish under variation of parameter ρ. E_3. The geometry of system S is then completely altered[15]. This kind of point is improbable insofar as only very particular settings of preference and technology parameters lead to totally unstable degenerate equilibrium. Even if it is difficult to ascribe proper values proper to create such a situation, we can nevertheless note that the probability of such an equilibrium is more likely in an economy condition with a stock of wealth relatively low; i.e. $a \in [0, \tilde{a}]$. In this interval the necessary condition for the existence of degenerate equilibrium is verified the slope function is such as: $\frac{\partial \psi}{\partial a} < 0$. This last condition implies that: $f_k + (f_m - f_k)m_a - c_a - n < 0$. Several elements may have an influence the sign of the relation. A high rate of population is a first possible factor. Variation of consumption in relation to the increase of wealth is a second factor. Consumption build up is detrimental to the accumulation of wealth. Lastly the high level of physical capital leads to a very low marginal productivity of capital so that the relation becomes negative. This interpretation implies that the share of money in wealth is lower than that of capital. However this condition is not verified since the

[15] See Hubbard and West (1997).

productivity of capital is higher than that of money; i.e. $\left(f_m - f_k\right) < 0$. In conclusion, a relatively low stock of wealth with a relatively low level of physical capital linked to a high rate of demography make the existence of a degenerate equilibrium more probable. It appears that in general it is possible to construct curves $\phi\left(\theta, a\right) = 0$ and $\psi\left(\theta, a\right) = 0$ for which there will exist an arbitrarily large odd number of steady states when ψ is a non-monotonic function. The exact structure of the dynamics depends on the properties of $\left(d\phi/da\right)\big|_{a_0}$, and $\left(d\psi/da\right)\big|_{a_0}$. Thus, the number of equilibrium points depends on the set of functions $\{f_{ka}, f_{k\theta}, f_{ma}, f_{km\theta}, c_{aa}, \ldots\}$ which is difficult to justify on the basis of economic theory[16]. More particularly, the existence of wealth effects can generate multiple equilibria. This is a limit to economic development through the central planning Kurz (1968) : *« It is clear that the presence of wealth effects can lead to any number of such stationary points; hence may lead to rather « non-ambitious » optimal strategy of economic growth »*, p. 354. More recently, the existence of multiple steady states equilibria gave rise to an important discussion on threshold effects like the "underdevelopment-trap" (cf. Azariadis and Drazen (1990)). In this contribution, many configurations are possible. The threshold effect can be significantly different from the concept of the "underdevelopment-trap" *à la* Azariadis-Drazen. For example in Figure 4, the relatively low steady state point, E_1, is locally unstable while this steady state is locally stable in Azariadis-Drazen. For those authors the lowest steady state can be thought of as development trap attracting all trajectories that start with a sufficiently small stock of wealth. But this configuration is also possible. As is seen in Figure 5, the unstable equilibrium, E_2, has an important "critical" meaning. There is a threshold region which contains an unstable steady state. The critical equilibrium divides the space in two regions. Each region includes one stable steady state. If the initial amount of money is such that the following relation $a < \overline{a}_2$ is checked, then any paths with initial conditions lying on this stable branch converge to the relatively low-level steady state E_1. In other words, there is an initial quantity of money for which the economy is trapped on a relatively low-level steady state. If such a case seems unlikely, it appears that a critical quantity of money should be determined by the monetary

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[16] It is well-known that the sign of the function is not easy to justify. See Kurz (1968), Blanchard and Fisher (1989), and Orphanides and Solow (1990).

authority in order to set the economy on the stable arm ; there must be a discontinuous jump in *m(0)* to the "correct" value leading the economy to a relatively high steady state. However, if the government is involved in setting the initial supply of money in such a way that the relationship $a < \overline{a}_2$ is verified, then the economy converges on the stable branch to the high steady state E_3. The equilibrium is a saddle-point and the stability property prevails if one only if the initial starting point lies exactly on the stable branch of the saddle-point. Is there reason to suppose that the initial condition always holds? We assume as indicated earlier that if *m(0)* and *k(0)* do not lie on the stable arm, assume that there must be a discontinuous jump in *m(0)* and *k(0)* to the "correct" value. This analysis shows that the monetary authority produces an unanticipated once-and-for-all jump in the money supply. This suggests that the system will tend to return to the relatively high-level equilibrium. Finally, it should be specifically interesting to examine if there are non-linearities in the relationship between capital level and the quantity of money. To do so, it should be useful to implement Hansen's (2000) endogenous threshold methodology to search for multiple regimes. Hansen develops a statistical theory of threshold estimation in the regression context that allows for cross-section observations. Using various proxies for money balances as potential threshold variables it should be interesting to show that countries are clustered in regimes that obey different growth paths and thus give direct evidence of multiple equilibria.

5 - Conclusion

This study addresses an issue that the technical barriers have led the authors as Orphanides and Solow (1990) or Brock (1974) to conjecture the presence of multiple equilibria under the assumption of complementarity of the utility function. As for Blanchard and Fisher (1989), they argue that the complementarity is still an open question. The objective is to complete this contribution in the work on the integration of money in a model of neo-classical growth to the "Sidrauski-Brock." The introduction of money in the utility function and the production function is justified on the ground of market transactions of goods and services, labour and capital. Furthermore, the complementarity of money with consumption goods in the utility function reflects the services rendered by the monetary asset in transactions. It is clear from this study that contrary to the findings of Brock, complementarity between money and consumption goods does not necessarily mean the presence of multiple equilibria when money is integrated into the production function. We show that in a model of economic growth there is a single steady-state where money has a positive value when the subjective discount rate of the representative agent is relatively high. The nature of unique equilibrium depends on the properties of utility functions and production and we have identified two cases. First, the study of the local state shows that the equilibrium can be a saddle-point as in most monetary models in two dimensions. The second case is a degenerate equilibrium when the saddle-point and repeller point are combined. The condition on the parameter of intertemporal preferences of the representative agent is a necessary but not sufficient to ensure the uniqueness of the equilibrium. We find also multiple equilibria. In this model, the agent's preference parameter may take the values that are at least two or three equilibria according to the function of utility and production function. The existence of multiple equilibria with money reveals a threshold effect significantly different from the concept of "underdevelopment-trap" modelled by Azariadis and Drazen (1990). In other words, there is an initial amount of money, as the economy can be directly blocked relatively low on the steady-state. The amount of money in circulation in the economy is critical in that it puts the dynamics of the economy on the stable branch leading to the high state or it sets the economy directly on the highest steady-state.

Appendix

To start, let us differentiate totally the system of the short term equilibrium:

$$u_{cc}dc + u_{cm}dm = d\theta\,,$$

$$u_{mc}dc + Bdm + Adk = -(f_m - f_k)d\theta\,,$$

$$dm + dk = da\,,$$

where,

$$A = (f_{mk} - f_{kk})\theta > 0\,,$$

$$B = u_{mm} + \theta\,(f_{mm} - f_{km}) < 0$$

We consider the following matrix:

$$\begin{bmatrix} u_{cc} & u_{cm} & 0 \\ u_{cm} & \mathrm{B} & \mathrm{A} \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} dc \\ dm \\ dk \end{bmatrix} = \begin{bmatrix} d\theta \\ (f_k - f_m)d\theta \\ da \end{bmatrix} \qquad \text{(I)}$$

The matrix of coefficients has the following determinant:

$$\Delta = u_{cc}\left(u_{mm} + \theta\ \left(f_{mm} - f_{km}\right) - \theta\ \left(f_{mk} - f_{kk}\right)\right) - u_{cm}^2,$$

$$\Delta = u_{cc}u_{mm} - u_{cm}^2 + u_{cc}\theta\ \left(f_{mm} + f_{kk} - 2\, f_{km}\right) > 0.$$

By assumption, the utility function is strictly concave; thus: $u_{cc}u_{mm} - u_{cm}^2 > 0$. We make assumptions: $f_{mm} - f_{km} < 0$, and $f_{kk} - f_{km} < 0$.

This dominant diagonal assumption seems reasonable. Despite the fact that $f_{mm} < 0$, we assume that money and physical capital are complements ($f_{km} > 0$). We can solve for all the partial derivatives of the derived demand equations. For example, the sign of the partial derivative with respect to θ is determined by setting da = 0 in (I):

$$\begin{bmatrix} u_{cc} & u_{cm} & 0 \\ u_{mc} & \mathrm{B} & \mathrm{A} \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{dc}{d\theta} \\ \dfrac{dm}{d\theta} \\ \dfrac{dk}{d\theta} \end{bmatrix} = \begin{bmatrix} d\theta \\ \left(f_k - f_m\right)d\theta \\ 0 \end{bmatrix}$$

The partial derivative of the consumption with respect to θ is:

$$\Delta_1 = \left(B - A\right) - u_{cm}\left(f_k - f_m\right) < 0, \text{ with } \left(f_k - f_m\right) > 0, \ \frac{\partial\, c}{\partial\, \theta} = \frac{\Delta_1}{\Delta} < 0.$$

In the same way, we calculate Δ_2, and Δ_3. We find the partial derivative for others variables with respect to θ : $\dfrac{\partial\, k}{\partial\, \theta} = \dfrac{\Delta_2}{\Delta} > 0$, $\dfrac{\partial\, m}{\partial\, \theta} = \dfrac{\Delta_3}{\Delta} < 0$.

The sign of partial derivatives with respect to θ are determined by setting da = 0. We obtain: $\dfrac{\partial\, c}{\partial a} = \dfrac{\Delta_4}{\Delta} > 0$, $\dfrac{\partial\, k}{\partial a} = \dfrac{\Delta_5}{\Delta} > 0$, and $\dfrac{\partial\, m}{\partial a} = \dfrac{\Delta_6}{\Delta} > 0$.

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