The Two-Parameter Long-Horizon Value-at-Risk

Guy Kaplanski[*1]

Haim Levy[2]

Abstract

Value-at-Risk (VaR) has become a standard measure for risk management and regulation. In the case of a two-parameter distribution, a common method among practitioners is first to calculate the daily VaR and then to apply it to a longer investment horizon by using the Square Root Rule (SRR). We show that the SRR is theoretically incorrect and propose a correct measure. The error from employing the SRR is positive for short horizons, inducing an overestimation of the true VaR, and negative for longer horizons, inducing underestimation of the true VaR. This error is relatively small for conservative portfolios and for short horizons. However, for risky portfolios and for long horizons – where accurate VaR is most important – the underestimation error is both substantial and systematic.

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[1] Bar-Ilan University, Ramat Gan. Tel: (972) 502262962; Fax: (972) 153502262962; Email: guykap@mail.biu.ac.il.

[2] The Hebrew University of Jerusalem and the Center of Law and Business, Ramat Gan.
Tel: (972) 258831011; Fax: (972) 25881341; Email: mshlevy@mscc.huji.ac.il.

1 - Introduction

Value-at-Risk (VaR) has become the standard measure for risk management and regulation.[1] VaR is extensively used by Basel (2006) for bank regulations (e.g. market risk regulations, credit risk mitigation techniques, and internal models for equity exposures), to constrain and control the risk exposure of the invested portfolio, report market risk exposure, and for many other uses. VaR may be calculated in numerous ways, each of which has its own specific advantages and disadvantages.[2] Sophisticated, large banks employ historical simulations to estimate VaR. Yet, the simplest and common method employed in practice, is the two-parameter VaR which implicitly assumes a two-parameter distribution. Specifically, when the distribution of the portfolio's return can be approximated by a two-parameter distribution, VaR can be calculated by using the mean and variance of the corresponding distribution. The most common procedure among practitioners is to calculate the daily VaR and then use the Square Root Rule (SRR) to extrapolate the longer horizon, e.g. a 10-day VaR (see Basel, 2006). Despite the simplicity of the SRR, and perhaps because of it, this approach may lead to large methodological errors – even when using a two-parameter distribution. The purpose of this paper is twofold:

1) First, we present the theoretically correct two-parameter VaR extrapolation formula and explain why use of the SRR is incorrect.
2) As the SRR is commonly employed in VaR calculations, rather than the correct formula, we next analyze the errors induced from employing the SRR.

We show that although employing the SRR is theoretically incorrect for time extrapolation, it does yield a reasonable VaR approximation for conservative portfolios and for short horizons, as the errors involved are relatively small. However, for risky portfolios and for long horizons – where accurate VaR is most important – the potential bias and the corresponding underestimation error are relatively large; moreover, the riskier the portfolio, the larger the underestimation error. Hence, employing the SRR in this case

[1] An excellent introduction and overview of VaR methodology can be found in Duffie and Pan (1997), Dowd and Blake (2006) and in the book by Jorion (2007).
[2] For a discussion on VaR general properties as a measure of risk, see e.g. Artzner, Delbaen, Eber, and Heath (1999), McNeil, Frey and Embrechts (2005), Daníelsson, Jorgensen, Samorodnitsky, Sarma, and de Vries (2005), Dhaene, Laeven, Vanduffel, Darkiewicz and Goovaerts, (2008), and Embrechts, Neslehova and Wüthrich (2009).

encourages risk-taking and thereby undermines the VaR regulation methodology.

Several studies have analyzed various aspects of VaR time extrapolation. Blake, Cairns and Dowd (2000) showed that employing the SRR when ignoring the mean in VaR calculations is incorrect – even for very short periods of time, such as a few days. In a complementary study, Cairns, Blake and Dowd (2003) analyzed the case of a continuous time model and showed that the direct impact of the mean on the horizon VaR is large and positive; hence, neglecting it induces a large overestimation error. Therefore, forecasting the mean is very important and may be a source for substantial error. In this study, we consider the various definitions of VaR, relative to the mean (for discussions on this issue, see, Kupiec, 1999, 2004), and suggest how the SRR should be employed in order to incorporate the direct impact of the mean. However, we show that in addition to this direct positive bias, there is also a less familiar indirect negative bias of the mean on the horizon variance, which induces an underestimation error. Moreover, as the two biases have opposite signs, the use of the SRR is not only inaccurate but does not enable predicting the direction of the error.

Other studies analyze the errors induced by the employment of the SRR when distributions are not *identical independent distributions* (i.i.d.). Daníelsson and Zigrand (2006) showed that when returns follow a stochastic jump diffusion process, employing the SRR induces a downward bias in risk estimation. They also point out that any time-scaling of volatilities requires returns to be conditionally homoscescedastic and conditionally serially uncorrelated – a condition which is almost as strong as i.i.d. Dorst and Nijman (1993) and Christoffersen, Diebold and Schuermann (1998) showed that when distributions are not i.i.d., the SRR is incorrect. A different approach to the time horizon VaR can be found in Fusai and Luciano (2001), who compared the static VaR - which is calculated by assuming that the portfolio composition is constant over time - with the dynamic VaR, which is calculated by assuming that the portfolio is adjusted dynamically over time. Using simulations, they show that the dynamic VaR is larger than the static VaR, even when distributions are normal and the portfolio adjustment costs nothing.

Finally, there are additional theoretical aspects of time horizon VaR. According to Kupiec (2004), for example, when VaR is used to determine the "economic capital", it should also account for the time value of money which can be achieved by augmenting it by the interest compensation. Hence, we stress at the outset that there may be additional problems involved in VaR time extrapolation; for example, if there is a serial correlation between daily observations, or volatility changes over time. Thus, in light of these and perhaps other problems, it is not surprising that large banks usually employ more complex methods to calculate their portfolios' VaR, which are based, for example, on historical simulations. Nevertheless, there are still many situations where the two-parameter distribution can be reasonably assumed. In these simplified cases, the two-parameter VaR, on which we focus in this study, provides a very simple and straightforward methodology. However, even in these cases, employing the SRR incorrectly may lead to substantial errors in VaR estimation. Moreover, these errors are methodological and occur even when the composition of the portfolio is constant over time and the distribution is a two-parameter i.i.d. Therefore, we propose a method for correcting these errors in the two-parameter case.

The remainder of the paper is organized as follows: Section 2 presents the theoretical background of a two-parameter distribution time extrapolation in comparison to the SRR mythology. Section 3 develops the multi-period VaR and analyzes the errors induced by employing the SRR. Section 4 illustrates the results using a numerical analysis. Section 5 concludes the paper.

2 - The horizon mean and variance and the SRR methodology

According to the current Basel regulations, market risk exposure is usually measured over a period of 10 trading days; in the case of credit risk a longer horizon may be applied.[3] Moreover, VaR is also used internally for longer horizons, such as 60 trading days – a case which, according to Hendricks and Hirtle (1997), represents the typical horizon the regulation should apply to. Moreover, VaR is often used for a longer time period, such as a full year (see, for example, Hendricks and Hirtle, 1997; Kupiec, 2002). In the two-parameter model, the two main elements required for calculating the two-parameter horizon VaR are *the* horizon *mean* and *variance*. In principle, one can directly calculate the relevant horizon mean and variance (and the corresponding VaR), thereby avoiding any potential time extrapolation errors. However, working with daily data (or even intra day data) is very common, as

[3] For example, according to Basel (2006), the horizon for calculating the haircuts on credit capital requirements due to credit collaterals is 20 trading days.

it offers important advantages – in particular, an increase in the number of contemporary observations which improve the statistical inference.[4] In what follows, we explain how to calculate the horizon mean and variance from the daily data parameters.

Denoting the one-period (e.g. one day) return in period t by $1 + R_t$, the multi-period (e.g. n-days) return is given by $1 + R_n$, where,

$$1 + R_n = \prod_{t=1}^{n}(1 + R_t)\,. \tag{1}$$

The n-days horizon mean, μ_n, is given by:

$$1 + \mu_n = (1 + \mu_1)^n\,, \tag{2}$$

where μ_1 is the one-period mean. A common procedure when calculating the horizon variance is as follows: first the log-return, $\log(P_t / P_{t-1})$, is calculated, after which the one-period variance, $\hat{\sigma}_1^2$, is calculated (where ^ indicates that this is the variance corresponding to the log-returns, rather than to the returns). Thus, using the log-returns variance (geometric mean), we obtain the SRR formula:

$$\hat{\sigma}_n^2 = n\hat{\sigma}_1^2\,. \tag{3}$$

This formula is a linear function in the horizon n.[5] In the financial literature, it is common to employ Eq. (3), and in particular in the calculation of VaR. Recall, however, that $\hat{\sigma}_1^2$ is calculated with log-returns rather than with returns. Alternatively, one can calculate the one-period variance of

[4] Although in the case of a direct calculation of the relevant horizon VaR a relatively large number of observations can also be achieved by looking far back into the past, this approach may involve less relevant observations, as market conditions – and in particular market volatility – may substantially change over time.

[5] We have $W_n = W_0 \times \prod(1 + R_t)$; hence, $\log(W_n) = \log(W_0) + \sum \log(1 + R_t)$,and with the i.i.d assumption we obtain $\hat{\sigma}(W_n) = \sqrt{n}\hat{\sigma}$, where $\hat{\sigma}$ is the one period standard deviation of the log-returns.

returns as σ_1^2, where the returns are given by (P_t / P_{t-1}). It is straightforward to show (see Tobin 1965 and Levy, 1972, 1973) that when using the variance of returns (arithmetic mean), rather than the variance of log-returns, the *n*-period horizon variance, σ_n^2, is given by the following non-linear formula:

$$\sigma_n^2 = \left[\sigma_1^2 + (1+\mu_1)^2\right]^n - (1+\mu_1)^{2n}, \tag{4}$$

where μ_1 and σ_1^2 are the one-period mean and variance. Eq. (4) has been suggested by Tobin (1965) and Levy (1972, 1973) to measure the *n*-period horizon variance. Another example presented by Ibbotson Associate (2006) – in a book which provides ample financial data corresponding to various horizons – employs eq. (4) to switch between estimates of parameters corresponding to various horizons. Thus, we have two time-extrapolating formulas given by Eqs. (3) and (4): the linear (SRR) formula and the non-linear formula. However, the correct formula is given by Eq. (4) as, unlike Eq. (3), in Eq. (4) returns and not log-returns are employed. While in VaR calculations log-returns are commonly employed, there is no theoretical justification for doing so.[6]

As Eq. (3) is widely employed, a natural question which emerges is: Under what conditions can Eq. (3) be theoretically justified? Indeed, there is one specific case where Eq. (3) measures the correct risk index: If the distribution under consideration is lognormal, the variance of log-returns, rather than the variance of returns, measures the prospect's risk. Namely, for a portfolio choice with an end-of-period lognormal distribution, $\sqrt{n}\hat{\sigma}$, is the appropriate risk index (for a proof, see Levy 1973). However, even in the important case of the lognormal distribution, eq. (3) cannot be employed for measuring VaR. Let us elaborate.

Attchison and Brown (1963) show that the variance of the lognormal distribution, σ^2, is given by,

[6] For example, Coval and Shumway (2001), who examine options' expected returns, advocate the employment of returns, rather than log-returns. Specifically, they assert that "…*log-scaling option returns may be quite problematic. Since log-transformation of any set of option returns over any finite holding-period will be significantly lower than the raw net returns. Put differently, the expected log return of any option held to expiration is negative infinity*."

$$\sigma^2 = e^{2\hat{\mu}+\hat{\sigma}^2}(e^{\hat{\sigma}^2}-1) = (1+\mu)^2 \times (e^{\hat{\sigma}^2}-1)\,, \tag{5}$$

where $\hat{\mu}$ and $\hat{\sigma}^2$ are the corresponding parameters of log-returns and μ is the mean return. Thus, to calculate VaR, one needs to employ the lognormal distribution quantile, rather than the normal distribution quantile, where the relationship between these two quantiles is given by,

$$Z_A(P) = e^{\hat{\mu}+Z(P)\hat{\sigma}}\,, \tag{6}$$

where $Z_A(P)$ and $Z(P)$ are the quantiles of the lognormal and normal distributions, respectively (see Attichson and Brown, 1963). Therefore, using $\hat{\sigma}$ to calculate $Z(P)$, which in turn yields VaR, is incorrect, since in the lognormal case $Z_A(P)$ is the relevant value corresponding to the probability P.

To sum up, on theoretical grounds, $\hat{\sigma}$ (and $\sqrt{n}\hat{\sigma}$) should not be employed for VaR calculations. This is because the employment of $\hat{\sigma}$ rather than σ yields an incorrect quantile value – even if distributions are indeed lognormal, let alone in the non-lognormal case – hence, an incorrect value of VaR is obtained. Therefore, Eq. (4) rather than Eq. (3) is the appropriate formula for VaR calculation.

3 - The investment horizon and VaR

In this section, we analyze the relationship between VaR and the investment horizon and explore the errors induced by employing the SRR (given by eq. (3)). Below, we use the following notations: VaR_1 denotes the one-period VaR, VaR_n denotes the n-period VaR, and VaR_n^e denotes the commonly employed SRR estimator of the n-period VaR, where the subscript e indicates (as we suggest in this study) that this measurement is involved with an error.

3.1 The definition of VaR

For a two-parameter distribution, we have $\Pr(R \le \mu_1 + \theta(\alpha)\sigma_1) = \alpha$, where as previously defined, μ_1 and σ_1 are the one-period mean and variance (in terms of returns) of *R*, and $\theta(\alpha)$ is negative value corresponding to the chosen confidence level, α, and to the assumed distribution. Namely, for a given two-parameter distribution, $\theta(\alpha)$ is only a function of the selected probability and does not depend on the distribution parameters. For example, for the normal distribution with a confidence level of $\alpha = 1\%$, $\theta(\alpha)$ is equal to about -2.33. The above probability can also be rewritten as,

$$\Pr(-R < -\mu_1 - \theta(\alpha)\sigma_1) = 1 - \alpha \, , \tag{8}$$

where for the relevant range for VaR calculations, $\alpha \in (0, {}^1/_2)$. Note that σ_1 in Eq. (8) is the standard deviation of returns, rather than of log-returns.

The standard industry definition of VaR is *the maximum loss corresponding to a given probability, α, over a given horizon* (see, for example, Duffie and Pan 1997 and Jorion, 2007). VaR can be defined in dollar terms or equivalently in terms of rate of return (i.e. for the current value of $1 of invested wealth). Without loss of generality, hereafter we employ the definition in terms of rate of return. Thus, VaR_1 is defined as,

$$\Pr(-R \le VaR_1(\alpha)) \equiv 1 - \alpha \, , \text{ where } \alpha \in (0,1/2). \tag{9}$$

Combining Eqs. (8) and (9), VaR_1 in the case of a two-parameter distribution is given by,

$$VaR_1(\alpha) = -\mu_1 - \theta(\alpha)\sigma_1 \, . \tag{10}$$

As $\theta(\alpha)$ is negative (in the range $\alpha \in (0, {}^1/_2)$), VaR_1 is defined in positive terms. For example, for the normal distribution with $\sigma_1 = 0.02$, $\mu_1 = 0$ and $\alpha = 1\%$, VaR_1 in terms of rate of return is equal to $0.02 \times 2.33 \cong 0.0466$. Hence, if the current value of the invested wealth is $100 million, VaR_1 in dollar terms is equal to $4.66 million.

The VaR in Eq. (10) is defined as the potential loss below the time horizon value (e.g., Alexander, 1998, p. 261) which also incorporates the expected return, μ_1. Namely, under this definition the positive value of μ_1 actually decreases VaR. Generally, when μ is the mean for short horizons, such as 10 trading days, the one-period mean, μ_1, is assumed to be zero. Moreover, VaR is frequently calculated from the current wealth (e.g. Basel, 2006); hence, it is implicitly assumed that $\mu = 0$. A justification for this approach is provided by Kupiec (2002, p.12-13) who correctly argues that while the definition in Eq. (10) estimates the so-called "unexpected losses," it will not produce accurate estimates of buffer stock capital requirements (for a further discussion on this VaR reference point, see Kupiec 1999, 2002, 2004). In this case – hereafter called the *Basel Case* – Eq. (10) is reduced to:

$$VaR_1 = -\theta(\alpha)\sigma_1 . \tag{10'}$$

As Eq.(10') is a specific case of Eq. (10), below we employ the general formula given by Eq. (10) to obtain more general results. Shifting from those general results to the specific case given by Eq. (10') is straightforward and is accomplished by simply assuming that $\mu = 0$, as is illustrated in the numerical analysis presented in Section 4. Let us now turn to the relationship between the one-period parameters and the *n*-period VaR.

3.2 The one-period parameters and the n-period VaR

The *n*-period expected mean and variance of *R*, assuming that *R* is i.i.d., are given by Eqs. (2) and (4), which reflect the compounding effect of the *n*-period parameters. As these parameters are used in VaR calculations, Eqs. (2) and (4) should be employed as inputs in the VaR_n calculations. However, in practice this may not be the case and the compounding effect is often ignored without theoretical justification. In particular, the SRR's common procedure to extrapolate the *n*-period VaR, yields the SRR estimator, VaR_n^e, given by,

$$VaR_n^e(\alpha) = VaR_1(\alpha)\sqrt{n} . \tag{11}$$

For example, Basel (2006) allows the use of the SRR for calculating the 10 trading days VaR, such that $VaR_{10}^e(\alpha) = VaR_1(\alpha)\sqrt{10}$. Combining Eqs. (10)

and (11) together with the fact that in practice firms usually employ the SRR on the log-returns yields:[7]

$$VaR_n^e(\alpha) = -\hat{\mu}_1\sqrt{n} - \theta(\alpha)\hat{\sigma}_1\sqrt{n} \, . \qquad (11')$$

Cairns, Blake and Dowd (2003) analyze the case of a continuous time model and show that the direct impact of the mean on the left hand side of Eq. (11') on the value of VaR is positive for the normal, student-*t,* and extreme-value distributions. Below we also incorporate the impact of the standard deviation, given on the left hand side of Eq. (11'), and show that the SRR approximation may lead to significant systematic biases from the value of the true VaR_n which tends to be negative for a long horizon, but is ambiguous for a relatively short horizon.

Substituting the *n*-period mean and variance from Eqs. (2) and (4) in Eq. (10) yields the correct value of VaR_n in terms of the one-period parameters:

$$VaR_n(\alpha) = 1 - (1+\mu_1)^n - \theta(\alpha)\sqrt{(\sigma_1^2 + (1+\mu_1)^2)^n - (1+\mu_1)^{2n}} \, . \qquad (12)$$

Note that if VaR is defined as *the loss relative to the current wealth* (the *Basel Case*), then the first expression, $1-(1+\mu_1)^n$, should be omitted. Eq. (12) provides the exact relationship between VaR_n and the one-period parameters in the case of a two-parameter i.i.d. Interestingly, VaR_n depends not only on the negative confidence level parameter, $\theta(\alpha)$, and on the one-period standard deviation, σ_1, but also on a complex relationship with the one-period mean, μ_1. Thus, the longer the horizon (larger *n*), the larger the *n*-period standard deviation, which in turn increases VaR_n. However, the longer the horizon, the larger the *n*-period mean as well, which decreases VaR_n. Hence, the sign of the impact of the increase in the horizon length on VaR_n is ambiguous. Although the various parameters have contradictory effects on VaR_n as *n* increases, the combined compounding effect for a sufficiently long period is

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[7] Note that employing $\hat{\sigma}_1$ and $\hat{\mu}_1$, rather than σ_1 and μ_1 in Eq. (11'), is another source of an error in addition to the errors induced from ignoring the compounding of returns. However, as we will show in Section 4, these errors are of the second order to the error induced from ignoring the compounding of returns. To be consistent, we assume that the firms that use the SRR consistently use it with log-returns.

always positive.[8] Thus, from a certain horizon, the positive effect of the standard deviation always exceeds the negative effect of the mean. Let us now turn to analyze these effects in more detail.

3.3 The error from employing the SRR

The error involved in the VaR_n measurement by using Eq. (3) rather than Eq. (4) is given by the difference between the SRR estimator, VaR_n^e, and the true value of VaR_n (Eqs. (11') and (12), respectively). This error, which is denoted by ΔVaR_n, is given by:

$$\begin{aligned}&\Delta VaR_n(\alpha) \equiv VaR_n^e(\alpha) - VaR_n(\alpha) = \\ &(1+\mu_1)^n - 1 - \hat{\mu}_1\sqrt{n} + \theta(\alpha)(\sqrt{(\sigma_1^2 + (1+\mu_1)^2)^n - (1+\mu_1)^{2n}} - \sqrt{n}\hat{\sigma}_1).\end{aligned} \tag{13}$$

To simplify the discussion below, let us also define $\Delta\mu \equiv (1+\mu_1)^n - 1 - \hat{\mu}_1\sqrt{n}$ and $\Delta\sigma \equiv \theta(\alpha)(\sqrt{(\sigma_1^2+(1+\mu_1)^2)^n-(1+\mu_1)^{2n}} - \sqrt{n}\hat{\sigma}_1)$ and rewrite Eq.(13) as follows:

$$\Delta VaR_n(\alpha) \equiv VaR_n^e(\alpha) - VaR_n(\alpha) \equiv \Delta\mu + \Delta\sigma\,. \tag{13'}$$

Eq. (13) presents the difference between the SRR estimator of VaR_n and its true value, where Eq. (13') allows us to relate to errors in measuring VaR_n to the mean bias, $\Delta\mu$, and the standard-deviation bias, $\Delta\sigma$. Obviously, if the one-

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[8] This can be seen by taking the limit of Eq. (12) as $n \to \infty$. Accordingly,

$$\begin{aligned}&\lim_{n\to\infty} VaR_n(\alpha) = \\ &\lim_{n\to\infty} 1-(1+\mu_1)^n - \theta(\alpha)\sqrt{\sigma_1^{2n} + n\sigma_1^{2(n-1)}(1+\mu_1)^2 + \ldots + n\sigma_1^2(1+\mu_1)^{2(n-1)}} > \\ &\lim_{n\to\infty} 1-(1+\mu_1)^n - \theta(\alpha)\sqrt{n\sigma_1^2(1+\mu_1)^{2(n-1)}} = \lim_{n\to\infty} 1-(1+\mu_1)^n - \theta(\alpha)\sqrt{n}\sigma_1(1+\mu_1)^{n-1} = \\ &\lim_{n\to\infty} 1-(1+\mu_1)^{n-1}(1+\mu_1+\theta(\alpha)\sqrt{n}\sigma_1) = +\infty.\end{aligned}$$

Note that in the inequality above, we replace the original expression by a smaller one (recall that $\theta(\alpha)<0$). Therefore, it is sufficient to prove that the smaller expression approaches $+\infty$. Indeed, the last term is positive for any $n > (1+\mu_1)^2/\sigma_1^2(\theta(\alpha))^2$ (as $\theta(\alpha)<0$). Hence, as $n\to\infty$ the whole term approaches $+\infty$.

period mean is zero, such as in the important *Basel Case*, $\Delta\mu = 0$. However, even in this case the total error, ΔVaR_n, is different from zero.[9]

In practice, the sign of $\Delta\mu$ is positive for every $n > 1$, and therefore always induces a positive error in measuring VaR_n. In contrast, $\Delta\sigma$ is negative for every $n > 1$, and therefore always induces a negative error in measuring VaR_n.[10] Another notable feature of the two biases is that both biases are unbounded as $\lim_{n\to\infty} \Delta\mu = +\infty$ and $\lim_{n\to\infty} \Delta\sigma = -\infty$.[11] From the above results, we see that $\Delta\mu$ is positive, inducing overestimation of VaR_n [see also

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[9] When $\mu_1 = 0$, we obtain $\sigma_n^2 = (\sigma_1^2 + 1)^n - 1 \neq n\hat{\sigma}_1^2$ from Eq. (4). Hence, although $\mu_1 = 0$, there are still two sources of errors: the compounding effect of the standard deviation and the employment of $\hat{\sigma}_1$ instead of σ_1.

[10] To prove that $\Delta\mu > 0$ for $n > 1$, we expand $\Delta\mu$ to a binomial series, canceling the +1 and –1, and bearing in mind that $\hat{\mu} < \mu$, which yields: $n\mu_1 + n(n-1)\mu_1^2/2!....+\mu_1^n - \hat{\mu}_1\sqrt{n} > \mu_1(n - \sqrt{n} + n(n-1)\mu_1^2/2!....+\mu_1^{n-1}) > 0$, for every $n > 1$. To prove that $\Delta\sigma > 0$ for $n > 1$, we bear in mind that $\theta(\alpha) < 0$ for $\alpha \in (0, {}^1/_2)$ and expand the two terms in brackets on the right-hand side of Eq. (13) (i.e. $\Delta\sigma$ apart from $\theta(\alpha)$) to a binomial series, which yields:

$$\sqrt{(\sigma_1^2 + (1+\mu_1)^2)^n - (1+\mu_1)^{2n}} - \sqrt{n}\hat{\sigma} > \sqrt{(\sigma^2+1)^n - 1} - \sqrt{n}\hat{\sigma} > \sqrt{(\sigma^2+1)^n - 1} - \sqrt{n}\sigma =$$

$$\sqrt{1 + n\sigma_1^2 + \frac{n(n-1)}{2!}\sigma_1^4 + \frac{n(n-1)(n-2)}{3!}\sigma_1^6 + ... - 1} - \sqrt{n\sigma_1^2} = \sqrt{n\sigma_1^2 + A} - \sqrt{n\sigma_1^2} > 0.$$

Defining all terms, apart from $n\sigma_1^2$, under the square root on the left-hand side of the equation by A, we see that $A > 0$, implying the inequality on the right-hand side. Note that in the first inequality we replace the original expression with a smaller one in two steps. First with a smaller expression under the square root and then by replacing $(-\hat{\sigma})$ by $(-\sigma)$ where $\hat{\sigma} < \sigma$. As the smaller expression is always positive, the original expression is also positive by necessity. Finally, as $\theta(\alpha) < 0$ in the range $\alpha \in (0, {}^1/_2)$, we have $\Delta\sigma < 0$ for every $n > 1$.

[11] This can be seen by using L'Hôpital's rule such that

$$\lim_{n\to\infty}(1+\mu_1)^n - 1 - \hat{\mu}_1\sqrt{n} = \lim_{n\to\infty} n(1+\mu_1)^{n-1} - \hat{\mu}_1/2\sqrt{n} = +\infty \text{ and}$$

$$\lim_{n\to\infty} \theta(\alpha)(\sqrt{(\sigma^2 + (1+\mu_1)^2)^n - (1+\mu_1)^{2n}} - \sqrt{n}\hat{\sigma}_1) =$$

$$\lim_{n\to\infty} \theta(\alpha)(\frac{(\sigma_1^2 + (1+\mu_1)^2)^n \ln((\sigma_1^2 + (1+\mu_1)^2) - (1+\mu_1)^{2n}\ln(1+\mu_1)^2}{2\sqrt{(\sigma_1^2 + (1+\mu_1)^2)^n - (1+\mu_1)^{2n}}} - \frac{1}{2\sqrt{n}}\hat{\sigma}_1) = -\infty.$$

Blake, Cairns and Dowd (2000)]. Less familiar is the effect of $\Delta\sigma$, which is negative, inducing underestimation of VaR_n. Therefore, $\Delta\mu$ is a matter of concern for institutions, as it may lead to over-regulation, while $\Delta\sigma$ is a matter of concern for the regulator since it may lead to under-regulation. Obviously, what matters to both parties is the combined effect of $\Delta\mu + \Delta\sigma$. As we shall see in the numerical example presented in Section 4, the sum of $\Delta\mu$ and $\Delta\sigma$ for a short horizon may be either negative or positive. However, for a sufficiently long horizon, it is always negative (this can also be seen from taking the limit of $\Delta\mu + \Delta\sigma$). Thus, the sign of the two biases together is ambiguous for short periods. However, for sufficiently long periods, the sign of the two effects together is always negative, inducing underestimation of the true VaR. More importantly, in the common case where VaR is defined as the loss relative to the current wealth (i.e. the *Basel Case*), the one-period mean is zero; hence, $\Delta\mu = 0$ and the error is always negative inducing to underestimation of the true VaR. In the next section, we illustrate the above results numerically.

4 - A Numerical analysis of two-parameter horizon VaR

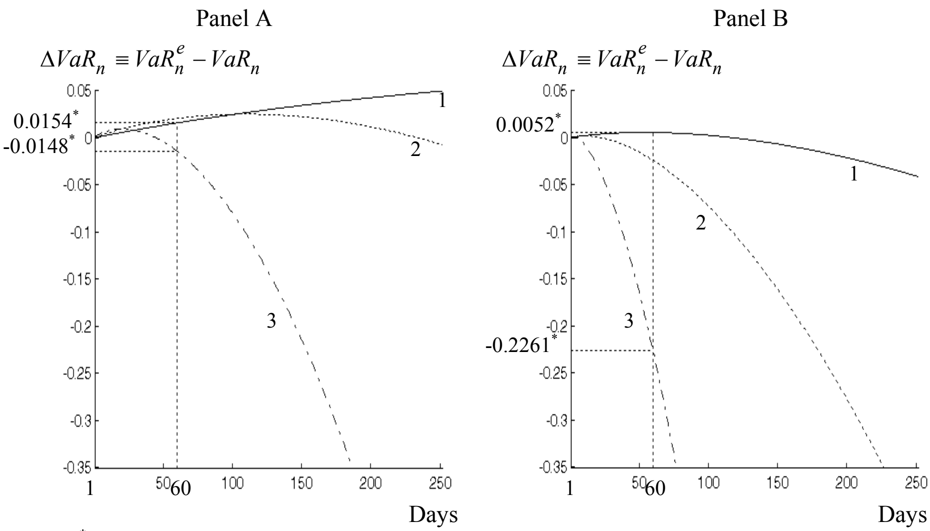
To further explore the induced error from employing the SRR, let us use a numerical analysis. First, some values for the one-period parameters μ_1 and σ_1 are needed. Therefore, the selected benchmark case corresponds to the U.S. market large stocks' returns.[12] For illustrative purposes, the other two cases correspond to riskier distributions with 2 and 4 times the benchmark daily standard deviation. These cases are fairly reasonable in the case of portfolios with large components of derivatives [see, for example, the parameters of SPX Call options in Coval and Shumway (2001)]. The time horizon ranges from 1 to 252 trading days.

[12] The assumed annually compounded expected mean in the benchmark case is 0.1 (a daily expected mean of $(1.1)^{1/252}-1 = 0.0003789$, see Eq. (2)) and the annually compounded standard deviation is 0.2 (a daily standard deviation of $\sqrt{[(0.2)^2 + (1.0003789)^{2\times252}]^{1/252} - (1.0003789)^2} = 0.0114$, see Eq. (4)).

4.1 The General Case

Figure 1 depicts the total error of the SRR estimator of VaR as a function of the horizon length, *n*, when VaR is calculated relative to the time horizon wealth (see Eq. (10)), i.e. when $\mu_1 > 0$ is taken into account.

Figure 1 **The total error received from using the SRR to estimate VaR_n when VaR is calculated relative to the time horizon wealth.**



*60 trading days error, $\Delta\ VaR_{60}$.

Key for the curves
1. The benchmark portfolio (the U.S. stock market parameters).
2. 2 times riskier portfolio than the benchmark portfolio.
3. 4 times riskier portfolio than the benchmark portfolio.

Panel A depicts the case of the normal distribution, while Panel B depicts the case of the student-*t* distribution with two degrees of freedom. The solid curves depict the benchmark case in which the parameters correspond approximately to the U.S. market large stocks' returns. The other two curves are for riskier distributions with 2 and 4 times the benchmark daily standard deviation.

Note that a positive error means that the SRR provides a higher VaR_n relative to the correct VaR_n (i.e. induces overestimation error), while the opposite holds true for a negative error. Panel A depicts the error corresponding to the normal distribution, while Panel B depicts the error corresponding to a student-*t* distribution with two degrees of freedom. In the benchmark normal case (solid curve) the error is positive, due to the positive bias, $\Delta\mu$. For example, the error after 60 days is +1.54%. However, in the case corresponding to the riskier normal distribution the error after 60 days is -1.48% and it drops an additional -20% after less than 150 days. An even more serious case, corresponding to the riskier student-*t* distribution, is reflected in the error after 60 days: -22.61%. To have a better resolution of the error magnitude, Table 1 reports the error for different horizons.

Table 1 **The total error (in %) from using the SRR to estimate VaR_n when VaR is calculated relative to the horizon wealth. The parameters of the three portfolios are presented in Figure 1.**

Days:	1*	2	10	30	60	100	150	200	250
Normal distribution									
Benchmark	0.01	0.03	0.25	0.79	1.54	2.43	3.40	4.22	4.90
2 times riskier	0.02	0.07	0.45	1.24	2.02	2.46	2.20	1.13	-0.70
4 times riskier	0.08	0.17	0.69	0.75	-1.48	-7.98	-21.40	-40.76	-66.45
Student-*t* distribution									
Benchmark	0.00	0.02	0.18	0.43	0.52	0.23	-0.70	-2.16	-4.10
2 times riskier	0.01	0.04	0.16	-0.31	-2.43	-7.29	-16.13	-27.76	-42.03
4 times riskier	0.04	0.05	-0.68	-6.49	-22.61	-55.49	-113.65	-191.06	-288.99

* Note that there is an error from the first day, due to the employment of $\hat{\sigma}_1$ and $\hat{\mu}_1$ rather than σ_1 and μ_1 in Eq. (11’). However, after only few days this error is of a second order to the error induced by the SRR.

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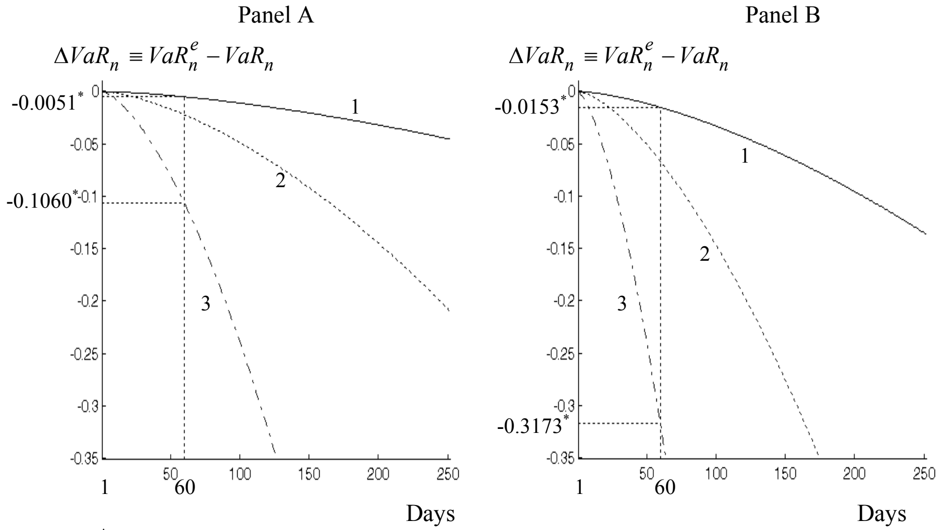
Figure 1 and Table 1 reveal that when the parameters correspond to a short horizon, and when the portfolio risk is similar to the U.S. equity market, the effect of $\Delta\mu$ on the error is dominant. In this case, the total error is relatively small and positive, inducing an overestimation of VaR_n. However, for riskier portfolios from a certain threshold horizon, the effect of $\Delta\sigma$ outweighs the effect of $\Delta\mu$; hence, the total error becomes negative and substantially large, inducing a substantial underestimation of VaR_n.

To sum up, the good news from the regulator's standpoint is that for the short horizon the SRR leads to overestimation of VaR_n. Naturally, from the economic perspective the unintentional additional safety driven by the VaR overestimation may not be optimal. A further encouraging result from the safety perspective is that if the risk of the position is reasonable, such as in the benchmark case, the horizon length for which the SRR leads to overestimation of VaR_n is quite long and can last several hundred days. In contrast, from a certain horizon threshold – which itself decreases with the risk of the portfolio – the SRR leads to underestimation of VaR_n. This error is very large for risky portfolios implying that the risk exposure is much larger than what is estimated by the SRR.

4.2 The Basel Case

The results are more dramatic in the realistic *Basel Case*, when VaR is measured relative to the current wealth (i.e. when $\Delta\mu = 0$). Figure 2 depicts the error induced from employing the SRR in this case.

Figure 2 **The total error from using the SRR to estimate VaR_n when VaR is calculated relative to the current wealth corresponding to Basel (2006).**



*60 trading days error, $\Delta\ VaR_{60}$.

Key for the curves

1. The benchmark portfolio (the U.S. stock market parameters).
2. 2 times riskier portfolio than the benchmark portfolio.
3. 4 times riskier portfolio than the benchmark portfolio.

Panel A depicts the case of the normal distribution, while Panel B depicts the case of the student-*t* distribution with two degrees of freedom. The parameters of the three portfolios are presented in Figure 1.

In this case, $\Delta\sigma$ is the only bias; hence, it is always negative, inducing an underestimation error. The error after 60 days corresponding to the benchmark (market parameters) normal distribution is small and just below zero. However, in the case corresponding to the riskier normal distribution, this error is -10.60% after 60 days, and -31.73% in the case corresponding to the student-*t* distribution. Note that in the case of Basel's (2006) market risk regulations these errors are further increased in terms of capital reserves, as they are multiplied by a factor ranging from 3 to 4. Table 2 reports the results of Figure 2 in more detail.

Table 2 **The total error (in %) from using the SRR to estimate VaR_n when VaR is calculated relative to the current wealth corresponding to Basel (2006). The parameters of the three portfolios are presented in Figure 2.**

Days:	1	2	10	30	60	100	150	200	250
Normal distribution									
Benchmark	0.00	0.00	-0.03	-0.18	-0.51	-1.11	-2.06	-3.20	-4.52
2 times riskier	0.00	-0.01	-0.15	-0.78	-2.23	-4.89	-9.20	-14.49	-20.73
4 times riskier	-0.02	-0.06	-0.68	-3.63	-10.60	-23.82	-46.27	-75.39	-111.62
Student-*t* distribution									
Benchmark	0.00	-0.01	-0.10	-0.54	-1.53	-3.32	-6.15	-9.58	-13.52
2 times riskier	-0.01	-0.04	-0.44	-2.33	-6.69	-14.65	-27.54	-43.38	-62.05
4 times riskier	-0.06	-0.18	-2.05	-10.86	-31.73	-71.33	-138.52	-225.69	-334.17

Figure 2 and Table 2 illustrate that when VaR is calculated relative to the current wealth, the SRR always leads to underestimation of VaR_n. The underestimation error significantly increases with both the time horizon and the portfolio risk. In other words, in the *Basel Case* the bias leads to a systematic methodological error, as the riskier the portfolio the larger the underestimating error. Hence, for risky portfolios the error is relatively large. Therefore, the use of the SRR in this case for risky portfolios encourages risk-taking, thereby undermining the VaR regulation policy.

5 - Concluding remarks

Value-at-Risk (VaR) is widely used and is often required by regulators. VaR can be calculated in various ways. Sophisticated banks employ simulations and statistical methods which take into account skewness and other properties of the distribution under consideration. Yet, a very common method is the simplified one, which assumes a two-parameter distribution. The most common way to calculate the *n*-period VaR, in the case of the two-parameter distribution, is to calculate the daily VaR, VaR_1, and to apply it to *n*-days horizon by the SRR, given by $VaR_n^e = VaR_1\sqrt{n}$, where VaR_n^e is the estimated *n*-period VaR (see Basel, 2006). However, this formula is incorrect for the horizon VaR calculation even when the distribution is a two-parameter distribution (e.g. normal) and i.i.d. To derive the correct *n*-period VaR corresponding to a two-parameter distribution, we employ Tobin's multi-period formulas to calculate the *n*-period mean and standard deviation of returns, which are used as inputs in the calculations of the *n*-period VaR, denoted by VaR_n. Now having both the correct formula and the incorrect, yet commonly employed formula, we analyze the errors resulting from the employment of the SRR, rather than the correct formula.

When VaR is calculated relative to the time horizon wealth, and when the parameters μ and σ correspond to the U.S. stock market, the error is positive for the short horizon, and becomes negative and large for a longer horizon. Thus, for the short horizon the SRR is conservative, inducing a larger estimated VaR than the true VaR. In contrast, for relatively long horizons the true VaR is larger than the measured one. The interesting result is that the riskier the portfolio (i.e. the higher μ and σ), the shorter the break-even horizon, n_0, such that for $n > n_0$ the error becomes negative. When VaR is calculated relative to the current wealth (the *Basel Case*) the error is always negative (i.e. for all $n > 1$ horizons), meaning that the true VaR is always larger than the estimated VaR. Moreover, this underestimating error is systematic; the riskier the specific portfolio, the larger the deviation from the true VaR.

The potential impact of the horizon length and the risk level of the specific portfolio on the size of the downward error in VaR calculation is a matter of concern, as the true risk exposure may be much larger than the measured and reported risk exposure. Accordingly, in the case of relatively long horizons and risky portfolios, a simple extrapolation of the multi-period VaR from the daily VaR is erroneous and should be completely avoided even for a two-parameter i.i.d.

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