First Passage and Excursion Time Models for Valuing Defaultable Bonds: a Review with Some Insights

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Abstract

In structural models of defaultable bond pricing default occurs at the first time a relevant process either reaches the default boundary or has spent continuously (or cumulatively) a fixed time period below that threshold. Unlike first-passage time approaches, excursion time models allow for a non-absorbing state of default. In this contribution, we provide a generalization of both first-passage and excursion time approaches by defining the default time as the first instant at which the firm value process (or another signaling process) either remains a certain time below the default threshold or hits a lower barrier. This corresponds, for instance, to a situation in which a firm is temporarily allowed to be short of funds, but enters default immediately when the financial distress becomes severe. We also examine the effects of different default time specifications on bond prices and credit spreads.

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1 - Introduction

Assessing default risk and the timing of such an event is crucial in order to evaluate the prices and returns of bonds. Although the effects of credit risk on bond values have long been analyzed, only relatively recently analytical models were developed in order to study and quantify these effects. A rational theory of credit risk is based on the pathbreaking model of Merton (1974). Since then, two main streams of research have been developed: structural models, based on a balance-sheet notion of solvency and option pricing theory, and reduced-form (or intensity-based) models of credit risk.

In this contribution, we study structural models of defaultable bond pricing in which the debtor's inability (or willingness) to pay is explicitly modeled. A firm defaults when it cannot meet its financial obligations. Structural approach is based in an explicit manner on the timing of such an event: default happens when assets are too low relative to liabilities, depending on some critical level being attained.

Çetin *et al.* (2004) argue that managers of a firm should know if default is about to occur. From managers' perspective, default is a predictable event. This perspective is also consistent with the formulation of structural credit risk models, in which the time of default is an *accessible* stopping time. On the other hand, the market has partial information[1], which makes default a surprise and the default time not predictable[2]. This latter perspective is consistent with reduced form credit risk models.

In the Black-Scholes-Merton model default may occur at the maturity date in the event that the issuer's assets are less than the face value of the debt. While first-passage models are based on the assumption that default occurs as soon as the asset value of the firm falls below a certain threshold. Such a boundary is often set equal to the face value of the firm's liabilities. Black and Cox (1976) introduce the idea that default can occur at the first time that assets drop to a sufficiently low level, either before or at the maturity of the debt. They assume a simple time-dependent default boundary.

First-passage time models have been widely studied and developed in the literature. A list of important contributions includes, but it is not limited to, Kim *et al.* (1993), Shimko *et al.* (1993), Longstaff and Schwartz (1995), Bryis and de Varenne (1997), Cathcart and El-Jahel (1998), Saá-Requejo and

[1] Or it has the managers' information set plus a noise (see Duffie and Lando, 2001), or leaves any knowledge of firm's value apart, by considering default as an exogenous event.
[2] The time of default becomes *inaccessible*, which means that one is not able to anticipate default.

Santa-Clara (1999), Taurén (1999), Collin-Dufresne and Goldstein (2001), Hsu *et al.* (2003), and Hui *et al.* (2003).

First-passage approach makes no distinction between the time at which a firm enters financial distress and the time at which it is either liquidated or a reorganization plan is accepted and the firm emerges from bankruptcy. In the classical approach, a firm defaults when its assets are too low according to some measure. Consequently the firm stops operating and is immediately liquidated. This definition of default seems not reflecting economic reality: bankruptcy laws often grant an extended time period for restructuring (see Eberhart *et al.*, 1999, and Helwege, 1999).

Recently there has been some interest in the credit risk literature in models in which the duration of the default (the time the firm spends in bankruptcy before being reorganized or liquidated) is explicitly modeled. Such models involve stopping times related to excursions and hence we will refer to as *excursion time models*. In the excursion approach the default time is defined as the first instant at which the process that governs default has spent continuously (or cumulatively) a fixed time period (called the window) below the default boundary. These models are based on the theory of valuation of Parisian options (see e.g. Chesney *et al.*, 1997, and Haber *et al.*, 1999) and occupation time derivatives (Hugonnier, 1999). Unlike first-passage time approach, the excursion time approach allows for a non-absorbing state of default. Moraux (2003) points out that only occupation time (cumulative excursion time) models describe the whole past financial distress. Such an approach considers the total time spent by the value process beyond the default threshold over the monitored time period, hence it seems more suitable in order to study the financial history and distress periods of the debt issuer.

Both the first-passage time and the Parisian time approaches can be generalized by defining the default time as the first instant at which the firm value process (or another signaling process) either remains a certain time below the default threshold or hits a lower barrier. This corresponds, for instance, to a real situation in which a firm is allowed temporarily to be short of funds, but enters default immediately when the financial distress becomes severe and a critical level, defined as a threshold set below the classical default boundary, is reached.

Based on such an idea, we propose a generalization of both the first-passage and excursion approaches, which takes into consideration the severity and persistency of the financial distress. The main focus is on the definition of the timing of default. Moreover, we examine the effects of different default time specifications on bond prices and credit spreads, under the assumptions of a particular structural model. To this aim, in the absence of closed form solutions, in general, numerical procedures are required, in particular when some simplifying assumptions regarding default-free interest rate modeling are relaxed. Monte Carlo simulation can be used in order to estimate the default probability.

An outline for this paper is as follows. Section 2 reviews first-passage time models and recalls some well known results. Excursion time approach is analyzed in section 3. In section 4, a new formulation of the timing of default is introduced. Section 5 presents some results of the simulation analysis. Section 6 concludes. In the appendix we summarize the model proposed by Cathcart and El-Jahel (1998), which will be used in the numerical experiments.

2 - First-passage time models

In the Black-Scholes-Merton model default may occur only at the maturity of the debt. In first-passage time models, introduced by Black and Cox (1976), this assumption is relaxed to allow for early default. More precisely, the time of default is defined as the first instant at which a relevant process, describing for instance the assets value of the firm, falls below a certain level called the default boundary (or default barrier). Such a boundary can be assumed as a constant threshold (as e.g. in the contribution by Longstaff and Schwartz, 1995), a time-dependent barrier (Black and Cox, 1976, consider an exponential default boundary), or it can be governed by some stochastic process (see, e.g., Saá-Requejo and Santa-Clara, 1999). A possible economic interpretation of the default boundary is the presence of some safety covenants in the contract. This boundary may be given exogenously or determined endogenously, as the solution of an optimal decision problem[3].

When the firm value falls below the distress threshold, the firm enters bankruptcy and may be either reorganized or liquidated. This point is crucial: in first-passage time modeling of credit risk no distinction is made between the time at which a firm enters bankruptcy (under Chapter 11) and the time at which it is either liquidated or the reorganization plan is accepted and the firm exits bankruptcy. In the excursion time approach, which will be discussed in section 3, the duration of the default (the time the firm spends in bankruptcy) is expressly modeled.

[3] See, among others, Leland (1994) and Leland and Toft (1996).

In this section, we take into consideration first-passage time models. Formally, let us consider a continuous time model on the time horizon $[0, \overline{T}]$. Assume that a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ is equipped with a filtration $(\mathfrak{F}_t; t \geq 0)$, provided some technical conditions, where $\mathfrak{F}_t$ represents the information set available at time *t*.

In a contingent claims framework (with full information), let us consider a firm with a single issue of debt in the form of a zero-coupon bond (hereafter ZCB) promising to pay the face value (which is assumed to be one euro) at the maturity date *T*. Let $v(t,T)$ denote the price at time $t \leq T$ of a defaultable ZCB with maturity *T*. Assume, moreover, that default-free zero-coupon bonds are traded for all maturities, and let *r* denote the default-free spot interest rate[4]. Both defaultable and default- free bonds markets are assumed to be arbitrage-free, so there exists an equivalent martingale measure $\mathbf{Q}$[5].

In this setting, consider a stochastic process $X = (X_t)_{t \geq 0}$ (continuous and adapted), which describes the firm's value (or another relevant firm's feature, as assets value, leverage ratio, operating cash flows, etc.), and the default boundary $H = (H_t)_{t \in [0,T]}$. Then, the information set at time *t* will be $\mathfrak{F}_t = \sigma(X_u; u \leq t)$. Note that, when the default boundary is modeled as a stochastic process, the information set is $\mathfrak{F}_t = \sigma(X_u, H_u; u \leq t)$.

As regards the recovery value upon default, different formulations are possible. In practice, when a firm enters default and is then liquidated, bondholders receive a variety of securities (different combinations of cash, zero-coupon or coupon bearing bonds, shares, warrants, and convertible securities). Here we make some simplifying assumptions (in line with Jarrow and Protter, 2004): bondholders may receive, for instance, the value of the boundary, which could be paid either at the default time or at maturity *T*. As an alternative, creditors may receive a fraction (which may be a parameter exogenously specified or even a random variable) of an equivalent default-free ZCB maturing at time *T*.

[4] For the moment, no assumption is made about the spot rate dynamics. *r* will be either a constant parameter, a time-varying function, or governed by some stochastic process.

[5] This probability measure may not be unique (see Jarrow and Protter, 2004).

In first-passage time models, the time of default is a random variable whose distribution is that of the first-hitting or first-passage time of the relevant process *X* through the default boundary $H = (H_t)_{t\in[0,T]}$. Formally,

$$\tau^H = \inf\{t \in [0,T) : X_t \le H_t\}, \tag{1}$$

with $\inf \varnothing = \infty$.

It turns out that, the random time τ^H is a predictable stopping time[6], which means that the time of default is not completely a surprise in first-passage time models, but can be anticipated in some sense by observing the trajectory taken by the process *X*. This feature will be exploited in the simulation analysis in order to estimate the average default time and the probability of default.

Of course, at maturity *T*, the default boundary has to be specified in a proper way to avoid inconsistencies (see Giesecke, 2004). At maturity in general default occurs if assets value is less than the face value of the debt, denoted by *F* (hence $H_T = F$). Let

$$\tau_T = \begin{cases} T & \text{if } X_T \le F \\ \infty & \text{otherwise;} \end{cases} \tag{2}$$

the default time, denoted by τ^*, is defined as the minimum between the first hitting time τ^H and the random time τ_T,

$$\tau^* = \tau^H \wedge \tau_T. \tag{3}$$

As an example, if we assume that the interest rate *r* is constant, and the recovery value upon default is equal to the barrier value at default, then the price at time *t* of a defaultable ZCB with maturity date *T* is given by

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[6] A stopping time τ is *predictable* if there exists a sequence of stopping times $(\tau_n)_{n\ge1}$ such that τ_n is increasing, $\tau_n < \tau$ on $\{\tau > 0\}$, for all *n*, and $\lim_{n\to\infty} \tau_n = \tau$ a.s. The sequence (τ_n) is said to announce τ. Jarrow and Protter (2004) point out that this property no longer holds, in general, when the process *X* presents jumps.

$$v(t,T) = \mathbf{E}_t \left(e^{-r(T-t)} \left(\mathbf{1}_{\{\tau^H \le T\}} H_{\tau^H} + \mathbf{1}_{\{\tau^H > T\}} 1 \right) \right), \tag{4}$$

where $v(t,T) = \mathbf{E}_t(\cdot)$ denotes expectation under the probability measure $\mathbf{Q}$, conditional to the information at time t (and conditional on no default until time t), and $\mathbf{1}_A$ is the indicator function of A.

$v(t,T)$ in equation (4) can be evaluated analytically in some special cases (see Bielecki and Rutkowski, 2002). In the absence of closed form solutions, one must resort to numerical approximations and computational techniques, in particular when some simplifying assumptions regarding default-free interest rate dynamics are relaxed. Monte Carlo simulation can be used in order to estimate the default probability (see Hsu *et al.*, 2003).

3 - Excursion and occupation time models

First-passage approach does not discriminate between the time at which a firm enters bankruptcy and the time at which it is either liquidated or reorganized: a firm defaults when its assets are too low according to some criterion, and is immediately liquidated. This definition of default no longer reflects economic reality: bankruptcy laws often grant an extended time period for restructuring[7]. In the excursion time approach, discussed in this section, the duration of the default is a crucial feature of the model.

Recently there has been some interest in the credit risk literature in models which involve stopping times related to excursions[8]. These models are based on the definitions of Parisian time and occupation time. The Parisian time can be defined as the first instant when a stochastic process (typically a Brownian motion) spends a given amount of time (called the window in the literature related to Parisian options) consecutively beyond a certain barrier. This excursion time has been first studied by Chesney *et al.* (1997) in order to define Parisian barrier options. Parisian options are financial derivatives for which the barrier in- or out-feature is activated when the process driving the

[7] Eberhart *et al.* (1999) investigate the stock return performance of a sample of firms emerging from bankruptcy, and find that the average time spent in bankruptcy (from the default announcement date through the first trading date after distress) is 22.39 months on average. See also Helwege (1999).

[8] See Moraux (2003), Galai *et al.* (2003), Paseka (2003), Broadie *et al.* (2004), François and Morellec (2004), Giesecke (2004), Yu *et al.* (2004).

underlying price spends continuously a certain amount of time beyond the barrier.

Unlike first-passage time approach, excursion time approach allows for a non-absorbing state of default. Excursion times seem then an attractive tool to define default that occurs when a “grace period” is granted. Moreover, compared to the definition of default time as a first-passage time, this formulation does not suffer from a threshold effect[9]. Even if the relevant process has crossed the barrier, default has not necessarily occurred yet. As a result, the model is more robust than in the “one-touch” case, in which the signaling process need only hit the barrier (no matter how shortly it stays below that threshold) for the default to occur.

Parisian times could then be useful in representing real situations when delays, between the default event and reorganization (or liquidation) of the firm, are involved. Such a delay could be either a fixed parameter exogenously specified (e.g. prescribed by a debt covenant), or a parameter of the model which has to be estimated from market data. As an alternative, the duration of the default (together with the default boundary) could also be the solution of an optimal decision problem, as in the model proposed by Paseka (2003).

In the following, we assume this delay is constant and it is denoted by Δ. For simplicity, let us consider a constant default boundary, *H*. Assume also $X_0 > H$.

Let

$$\tau_t^H = \sup\{s \le t : X_s = H\} \qquad (5)$$

be the last time, before *t*, that the process *X* hits the threshold *H*, and define the excursion (Parisian) time as follows,

$$\tau_\Delta^H = \inf\{t \ge 0 : (t - \tau_t^H)\mathbf{1}_{\{X_t \le H\}} \ge \Delta\}. \qquad (6)$$

The default time is defined as the random time τ_Δ^H and the default probability is $\mathbf{Q}[\tau_\Delta^H \le T] = \mathbf{E}[\mathbf{1}_{\{X_t \le H\}}]$.

Intuitively, the probability of default in the excursion time approach should fall between the first-passage time case and the default probability in the Black-Scholes-Merton model. When the time window Δ is zero, we

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[9] See Haber *et al.* (1999).

recover the simple first-passage case, while when Δ is very long and approaches *T*, one expects the probability of default to reduce to that in the Black-Scholes-Merton model. In this case, one will check for default only at maturity of the bond.

Let us consider the Parisian time case. Define the barrier time

$$\Delta_t = t - \tau_t^H \qquad t \in [0, T] \tag{7}$$

as the length of time the process *X* has spent below the default barrier in the current excursion. Δ_t is simply the difference between the current time *t* and the last time *X* hits the barrier. Note that, if the process *X* is currently above the default threshold, then $\Delta_t = 0$.

Starting from $X_0 > H$, the dynamics of Δ_t is described as follows[10]

$$d\Delta_t = \begin{cases} dt & X_t < H \\ -\Delta_{t^-} & X_t = H \\ 0 & X_t > H, \end{cases} \tag{8}$$

where Δ_{t^-} is the left limit of Δ_t. The "*clock*" Δ_t is reset to zero when the barrier is reached from below, and it does not change if the process is above the barrier. Default occurs only if the length of the current excursion exceeds a given level, $\Delta_t \geq \Delta$, where Δ is the barrier time parameter. The probability of default is $\mathbf{Q}[\tau_\Delta^H \leq T] = \mathbf{Q}[\Delta_t \geq \Delta]$. In the absence of closed-form solutions, the probability $\mathbf{Q}[\Delta_t \geq \Delta]$ can be computed by simulation[11]. By exploiting the definition of the clock's dynamics (8), the problem can be implemented in a very intuitive way.

As previously observed, the Parisian time approach does not suffer from a threshold effect, in that a simple hitting of the barrier does not cause default with immediate liquidation of the firm. The triggering of the barrier is

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[10] See Haber *et al.* (1999).

[11] It is well known that the effects of discretizeation of time can be important when dealing with the evaluation of barrier options. The same problems arise when we use Monte Carlo simulation to evaluate the default probability. Nevertheless, this method, even if typically slow and not so accurate, is flexible and easy to implement.

fairly robust, nevertheless, this approach suffers from another anomaly. As pointed out by Haber *et al.* (1999), the resetting of Δ_t is still sensitive to short-term movements of the value process *X*. For example, consider the case of a firm that enters bankruptcy and then, after the process *X* has spent a long time (but less than Δ) under the barrier, it reaches the default threshold (from below) again, before the grace period has elapsed. At the maturity of the bond, the firm will be able to repay its debt, even if it has spent in financial distress almost all the time. As an alternative example, consider a firm that enters and exists bankruptcy more than once during the life of the debt, this without causing liquidation of the firm (note that successive defaults are possible in the economic reality).

In order to cope with these anomalies, Haber *et al.* (1999) and Moraux (2003) suggest to consider the cumulative excursion (or occupation) time instead of the continuous excursion time[12]. Moraux (2003) points out that only occupation time models take into account the whole past financial distress of the firm, even if this approach may not be appropriate for very long term bonds. Such an approach considers the total time (over the monitored interval) spent by the value process beyond the default threshold, hence it seems more suitable in order to study the financial history and distress periods of the debt issuer.

Formally, if *H* is the barrier, *T* the maturity of the bond, and Δ is the maximum duration of time allowed below the barrier, the occupation time is defined as follows

$$\Delta_0^T = \int_0^T \mathbf{1}_{\{X_t \le H\}} dt. \tag{9}$$

Δ_0^T is a measure of the amount of time the process *X* spends below the barrier during the time interval $[0,T]$. The probability of default is defined by $\mathbf{Q}[\Delta_0^T \ge \Delta] = \mathbf{E}[\mathbf{1}_{\{\Delta_0^T \ge \Delta\}}]$.

In this formulation, the clock is not reset to zero, and the default time is defined as the first instant the process *X* has spent totally an amount of time Δ below the default boundary,

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[12] Options linked to occupation time are also called “ParAsian” options in Haber *et al.* (1999). See also Hugonnier (1999).

$$\tau^* = \inf\{t \geq 0 \colon \Delta_0^t \geq \Delta\}. \tag{10}$$

Monte Carlo simulation can be used in order to compute an estimate of the average time of default and the default probability.

It is worth noting that the first-passage approach is more favorable to bondholders, while the excursion time approach, allowing for a grace period, favors firm's rights. Occupation time approach falls, in a certain sense, between the classical first-passage and the Parisian time approaches.

4 - Excursions height and length: an application to credit risk

Gauthier (2002) studies the stopping time defined by the first instant when a Brownian motion either spends consecutively more than a certain time above a given level, or reaches another level. This stopping time generalizes both the simple hitting time and the Parisian time. Gauthier derives the Laplace transform of this stopping time and applies the obtained results to the valuation of investment projects with a delay constraint and the possibility of starting immediately the project but at a higher cost.

In this section, by exploiting Gauthier's idea, we define the default time as the first instant when a relevant process either stays continuously for a certain time interval under the default boundary, or hits another lower threshold. In such a way, we could model situations in which a firm is allowed temporarily to be short of funds, but enters default immediately when the financial distress becomes severe. If either the firm's negative cash flows persist for an extended period of time, or the firm exhausts all its lines of credit and liquid assets, it defaults right away.

More precisely, we are interested in the first time τ^* when the process $(X_t)_{t\geq 0}$ governing default either spends consecutively a given period of time Δ (with $\Delta < T$) below the default boundary *H*, or reaches the lower threshold $H^- = H$.

Let, as in the previous section,

$$\tau_t^H = \sup\{s \leq t : X_s = H\} \tag{11}$$

be the last time, before *t*, that the process *X* hits the barrier *H*. Define the excursion time

$$\tau_{\Delta}^{H} = \inf\{t \geq 0 : \Delta_t \mathbf{1}_{\{X_t \leq H\}} \geq \Delta\}, \tag{12}$$

where $\Delta_t = t - \tau_t^H$, and the hitting time

$$\tau^{H^-} = \inf\{t \geq 0 : X_t \leq H^-\}. \tag{13}$$

In this formulation, the default time is

$$\tau^* = \min(\tau_{\Delta}^{H}, \tau^{H^-}), \tag{14}$$

and the probability of default is $\mathbf{Q}[\tau^* \leq T] = \mathbf{E}[\mathbf{1}_{\{\tau^* \leq T\}}]$.

The definition of the stopping time τ^* in (14) extends both the excursion time and the first-passage time definitions. τ^* is the first time the process *X* leaves the region defined by the default boundary *H* on top and the lower boundary H^- on bottom, and length Δ, without reaching the barrier *H* again from below. Credit risk models based on excursions height and length account for both the severity (height) and persistency (length) of the financial distress (the excursion).

Let us consider, for simplicity, constant boundaries. Figure 1 shows an example of excursions in the *distress region*: default occurs only for path *A* at time τ_A.

As an alternative, one could also model *H* and H^- as time-varying barriers:

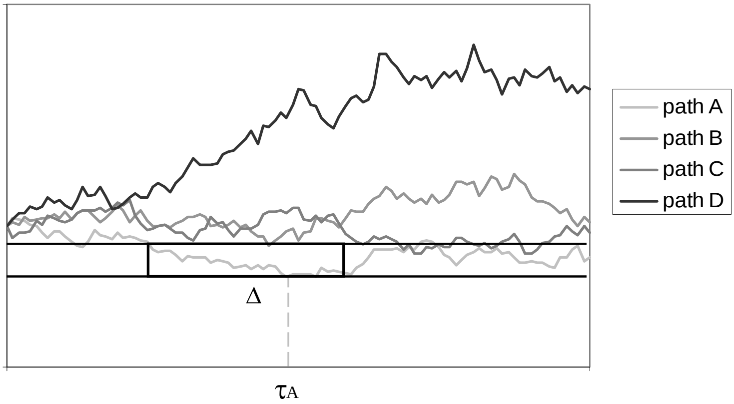
$$H_t = \overline{H} e^{-\gamma(T-t)} \tag{15}$$

and

$$H_t^- = \overline{H} e^{-\eta(T-t)}, \tag{16}$$

with parameters $\gamma < \eta$ (γ and η can be interpreted as discount rates), and $\overline{H} > 0$ (typically set equal to the face value of the debt *F*). In order to avoid inconsistencies, the event of default at maturity has to be defined carefully, as discussed in section 2.

Figure 1 Instances of default time in the approach based on excursions height and length.



These boundaries could also be endogenously determined, as in the model proposed by Paseka (2003) (which is similar to the approach here presented), in which default occurs at the time

$$\tau^B = \inf\{t > 0 : X_t = H^B\}, \tag{17}$$

where $H^B > 0$ is the bankruptcy barrier (the level at which the firm enters bankruptcy). Liquidation occurs only if the reorganization is unsuccessful, and this happens at time

$$\tau^L = \inf\{t > 0 : X_t = H^L\}, \tag{18}$$

where $H^L < H^B$ is the liquidation barrier. Otherwise, if the value of the firm rises to a certain level H^R (which represents the reorganization boundary) at which a reorganization plan is proposed by the debtor, the firm exits default. Define the random time

$$\tau^R = \inf\{t > 0 : X_t = H^R\}, \tag{19}$$

then the timing of exit from bankruptcy is the minimum between the reorganization time and the liquidation time

$$\tau^* = \tau^R \wedge \tau^L . \tag{20}$$

In Paseka's model, the level H^L is exogenously specified, while the boundaries H^B and H^R are both determined endogenously in order to maximize equity value. As a result, also the duration of bankruptcy is a solution of a bargaining process.

The idea of modeling excursions in space and time for valuing corporate debt could be generalized by introducing multiple boundaries at which the firm value process must take on specific values. In such a framework, the triggering of a barrier signals some credit event like, for example, a credit migration (a downgrade or an upgrade in rating of the issuer). To this regard, Esteghamat (2003) points out that “boundaries in the space separate credit events”, and proposes a model which connects structural and credit rating models by treating credit events as crossings of a multidimensional stochastic process through bounded regions in a probability space.

A further generalization of the approach based on excursions height and length, and related stopping times, is the idea of weighting large deviations below the default barrier more strongly, in order to take into account the severity of the financial distress. As suggested by Haber *et al.* (1999), in the modeling of financial options with Parisian features, the speed at which the clock process Δ_t changes could be proportional to the distance of the value process X from the default barrier. Another possible specification consists in considering upper boundaries together with lower boundaries: the time spent above the default threshold might be subtracted from the time spent below it. This latter formulation could be useful in modeling situations in which a firm alternates periods of fund shortages to excess of liquidity. One can easily envisage many other possible stylized situations.

5 - Analysis of some numerical examples

In this section we examine the effects of different default time specifications on bond prices and credit spreads under the assumptions of one of the structural models proposed in the literature. In particular, in the simulation analysis, we adopt the model proposed by Cathcart and El-Jahel (1998) which is summarized in the appendix.

In order to determine the default probability we used analytical formulas (when available) and Monte Carlo simulation in all other cases. As already observed, the idea of modeling the time of default as a hitting or excursion time is applied in a very easy and intuitive manner in the simulation. One generates a sufficiently high number of trajectories for the process X that signals default, and the default probability is simply approximated by the relative frequency of the default triggering event.

In the experiments carried out, we generated 100000 paths (50000 antithetic) of the process X. We considered a time step equal to $\Delta t = 1/250$ (which approximatively corresponds to daily observations of the process X) and bond maturities (in years) in the range [0, 20].

Figure 2 shows the bond prices (where p, represented by a dotted line, is the value of the default-free bond) and the term structure of credit spreads for different values of the writedown w and for different specifications of the default time. More precisely, in the first case shown in figure 2, we used the original model of Cathcart and El-Jahel (1998). In the second case reported, we introduce a delay of $\Delta = 0.5$ (six months), while in the third case we consider the cumulative excursion time. In the fourth case, we introduce both the grace period Δ and a lower boundary $H^{-} = 0.9H$.

The term structure of credit spreads displays the typical humped shape. We observe that, the introduction of a delay in the timing of default has the effect of changing the shape and level of credit spreads. When the default barrier is assumed constant, as in this case, and being other model factors unchanged, we obtain higher bond values and then lower credit spreads. This result is also intuitive. Observing the graphics in figure 2, it seems that different hypothesis made on the excursions have no such an effect on the spreads. This is, actually, due to fact that the process X has continuous paths and "takes some time" to reach the lower barrier H^{-} (which is set at 90% of the value of H). On the other hand, what changes is the expected time of default: the introduction of a grace period and different definitions of default time have the effect of avoiding or at least postponing default. Intuitively, the average time of default in excursion models is higher than in the first-passage case (the results of the simulation are not reported here in detail). Default occurs on average before in the cumulative excursion case than in the Parisian case. The introduction of a lower boundary causes earlier default and the probability of default is higher. Recall also the definition of default at maturity (which occurs in any case if the relevant process is below a certain level); such an assumption has an effect on the way we compute the probability of default.

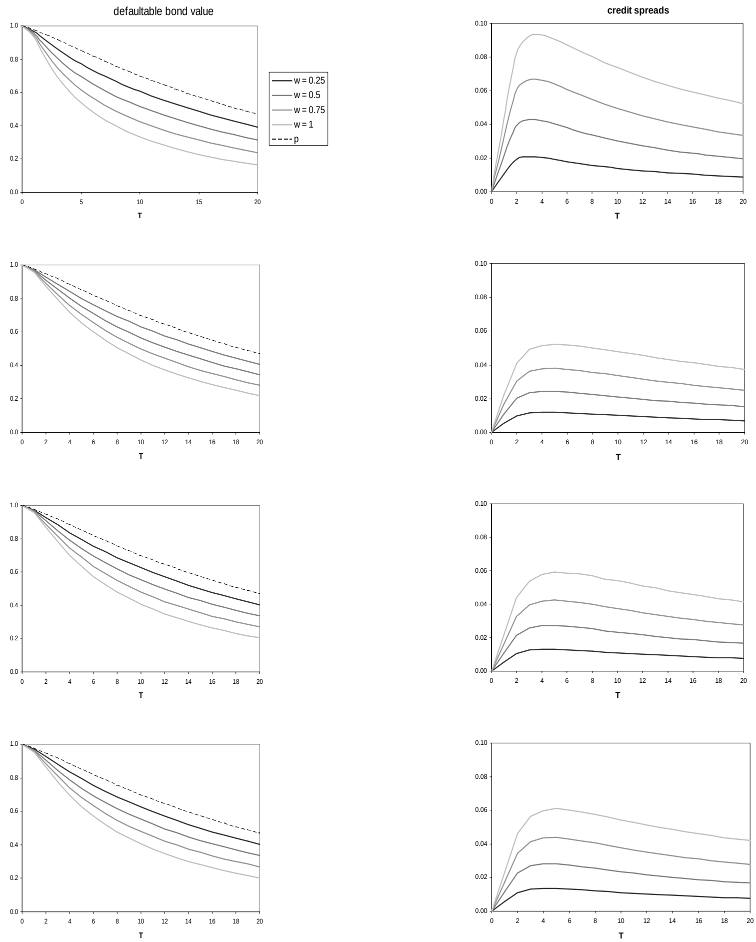
Figure 3 shows another similar experiment, in which we consider a higher ratio $X_0 / H = 2$ (only the results for the term structure of credit spreads are reported). Of course, in such a case the probability of default is lower. It is interesting to observe that the default probability (and consequently the spreads) are zero in the short period: this fact is typical in structural models and is due to the assumption of continuous paths for the process which drives default. In figure 4 we let the ratio X_0 / H vary and held other parameters constant: the lower the solvency ratio, the higher the default probability and consequently the credit spreads, with different effects on the timing of default.

6 - Conclusion

A variety of approaches have been proposed in the literature with the aim of evaluating defaultable debt and trying to better explain the process that drives default. First-passage approach makes no distinction between default and liquidation. Recently there has been increasing interest in the credit risk literature in models which involve stopping times related to excursions. This relatively new approach has been discussed, highlighting its advantages and disadvantages.

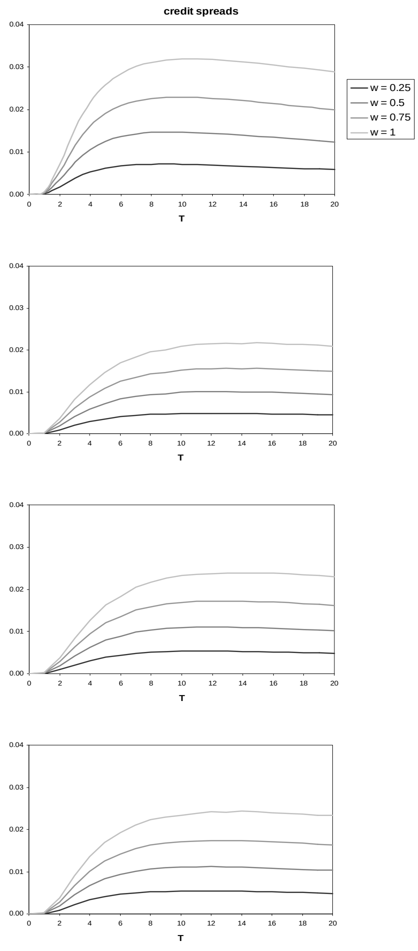
We then proposed a generalization of both the first-hitting time and the excursion time approaches, which consists in defining the default time as the first instant at which the firm value process (or another signaling process) either has spent a certain time below the default threshold or hits a lower barrier. In the present work, we focused more on the different definitions of the timing of default, rather than on other important and interesting issues (which require further research) like the specification of the default boundary, the dynamics of the process that governs default, the recovery value upon default, the dynamics of the default-free interest rate and its correlation with the process that signals default. On the other hand, the idea of modeling the timing of default as either a hitting or excursion time is potentially applicable to a very large class of models.

Figure 2 Defaultable ZCB and default-free ZCB prices and the term structure of credit spreads.



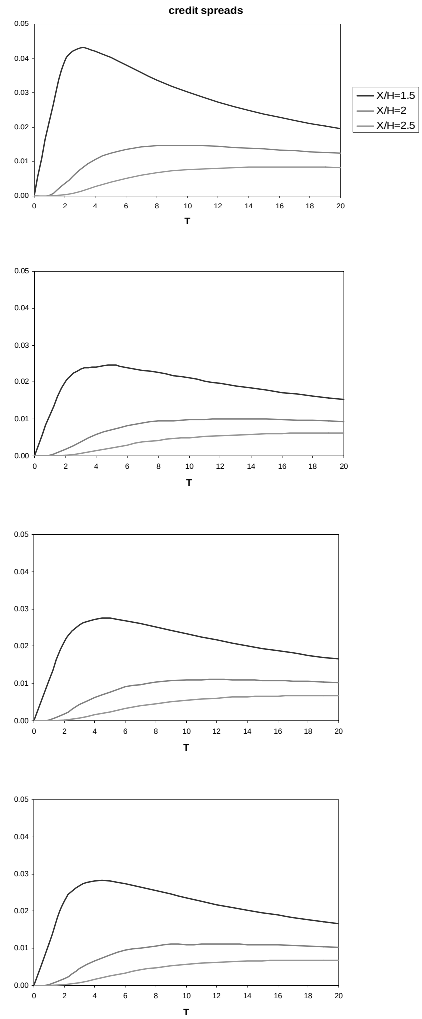
Defaultable ZCB and default-free ZCB (dotted line) prices and the term structure of credit spreads for different values of the writedown $w \in \{0.25, 0.5, 0.75, 1\}$, in the first-passage, excursion (with $\Delta = 0.5$), cumulative excursion, and excursion in height and length (with $H^- = 0.9H$) cases. The other parameters are: $\alpha = 0.02$, $\sigma_X = 0.2$, $X_0 / H = 1.5$, $r = 0.02$, $\theta = 0.04$, $\kappa = 0.5$, $\sigma_r = 0.03$.

Figure 3 The term structure of credit spreads.



The term structure of credit spreads for different values of the writedown $w \in \{0.25, 0.5, 0.75, 1\}$, in the first-passage, excursion (with $\Delta = 0.5$), cumulative excursion, and excursion in height and length (with $H^- = 0.9H$) cases. The other parameters are: $\alpha = 0.02$, $\sigma_X = 0.2$, $X_0 / H = 2$, $r = 0.02$, $\theta = 0.04$, $\kappa = 0.5$, $\sigma_r = 0.03$.

Figure 4 The term structure of credit spreads.



The term structure of credit spreads for different values of the ratio $X_0 / H \in \{1.5, 2, 2.5\}$, in the first-passage, excursion (with $\Delta = 0.5$), cumulative excursion, and excursion in height and length (with $H^- = 0.9H$) cases. The other parameters are: $\alpha = 0.02$, $\sigma_X = 0.2$, $w = 0.5$, $r = 0.02$, $\theta = 0.04$, $\kappa = 0.5$, $\sigma_r = 0.03$.

Finally, we examined the effects of different default time specifications on bond prices and credit spreads within a stylized model. Based on the results of the numerical analysis carried out, we observed that the introduction of a delay in the timing of default has an impact on credit spreads. When the default barrier is assumed constant, and being other model factors unchanged, we obtain (as expected) higher bond values and, hence, lower credit spreads.

Observing the term structure of credit spreads obtained in the numerical experiments, one can notice that at short maturities the issuer has no or very low default risk, after which, conditional on no earlier default, credit spreads rises (also significantly in some cases). Many authors point out that such term structures of default probabilities and credit spreads are not realistic. The excursion approach retains (and even stresses) this characteristic which is common to first-passage time models, based on a diffusion process that drives default, and perfect knowledge of the firm's value and default boundary. Even thought an issuer may be of poor credit
quality the probability of default within a short time horizon is very low. Models of firm dynamics that allow for jumps, or imperfect information (of course at the expense of some tractability), would generate term structure of credit spreads and default probabilities more in accord with observed data. This important issue is left for future research.

Appendix. A model with constant default boundary and stochastic default-free interest rate

The model proposed by Cathcart and El-Jahel (1998) is in line with that of Longstaff and Schwartz (1995). The authors introduce the idea of a *signaling process* whose dynamics governs default. More precisely, default occurs at the first instant such a signaling process hits a constant default threshold.

Assume a perfect frictionless market, and let $X = (X_t)_{t\geq 0}$ denote the risk-neutralized process that signals default, such that

$$dX_t = \alpha X_t dt + \sigma_X X_t dW_t^X \qquad X_0 > 0, \tag{21}$$

where α and $\sigma_X > 0$ are constant parameters, and W^X is a standard Wiener process.

Note that, in this setting, the assumption of a constant drift, not depending on *r*, is justified by the fact that the signaling process is not necessarily a value process. The signaling process should be able to describe a variety of effects that can determine default, like for example shortage of funds. The authors point out that this formulation is also suitable for sovereign institutions, for which it would be difficult to identify underlying assets.

The default-free spot rate *r* is assumed to follow a mean reverting square root process, according to Cox, Ingersoll and Ross (1985) model (CIR). Its risk-adjusted dynamics is

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t}\, dW_t^r \,, \tag{22}$$

where $r_0 > 0$, κ, θ and $\sigma_r > 0$ are constant parameters, and W^r is a standard Wiener process. The parameters κ and θ are the long-term mean value of *r* and the speed of adjustment to θ, respectively.

Let $p(t,T)$ denote the price at time *t* of a default-free ZCB with maturity date *T*. In CIR model, the value of the bond can be calculated using the formula

$$p(t,T) = A(\tau) e^{-B(\tau) r_t} \,, \tag{23}$$

where $\tau = T - t$,

$$A(\tau) = \left[\frac{2\gamma\, e^{(\gamma+\kappa)\frac{\tau}{2}}}{2\gamma + (\gamma+\kappa)(e^{\gamma\tau} - 1)} \right]^{2\kappa\theta/\sigma^2} \,, \tag{24}$$

$$B(\tau) = \frac{2\,(e^{\gamma\tau} - 1)}{2\gamma + (\gamma+\kappa)(e^{\gamma\tau} - 1)} \,, \tag{25}$$

and

$$\gamma = \sqrt{\kappa^2 + 2\sigma_r^2} \,. \tag{26}$$

Cathcart and El-Jahel (1998) assume, furthermore, that W^X and W^r are uncorrelated. As an immediate result, the process that governs default is independent of the interest rate process. The debt issuer enters default at the first instant the process *X* reaches the barrier *H*.

A crucial assumption concerns the recovery value upon default. Debt holders receive $1-w$ default-free ZCBs with the same maturity and face value of the corporate debt. The percentage of the *writedown* w is assumed constant. In particular, this assumption allows to valuing securities (even complex liabilities) independently from other liabilities issued by the firm. Coupon bearing bonds can be priced in an easy way, as a portfolio of ZCBs.

Let $v(X,r;t,T)$ denote the value at time *t* of a defaultable ZCB, with maturity *T*. *v* satisfies the partial differential equation

$$\alpha X \frac{\partial v}{\partial X} + \kappa(\theta - r)\frac{\partial v}{\partial r} + X^2 \frac{\sigma_X^2}{2}\frac{\partial^2 v}{\partial X^2} + r\frac{\sigma_r^2}{2}\frac{\partial^2 v}{\partial r^2} - \frac{\partial v}{\partial \tau} = rv\,, \quad (27)$$

subjected to some conditions (see Cathcart and El-Jahel, 1998). If default has not occurred, then

$$v(X,r;T,T) = 1. \quad (28)$$

The following boundary condition holds,

$$v(\infty, r;t,T) = p(t,T)\,. \quad (29)$$

In the event of default, $X_t = H$, one has

$$v(H,r;t,T) = (1-w)\,p(t,T)\,. \quad (30)$$

Moreover, the boundary conditions for *r* are

$$v(X,\infty;t,T) = 0, \qquad v(X,r;t,T) < \infty \qquad r \to 0\,. \quad (31)$$

Given the hypothesis of the model, the value of a defaultable ZCB is

$$v(X,r;t,T) = p(t,T) - w\,p(t,T)\,\mathbf{Q}_t(\tau^H \le T)\,, \quad (32)$$

where $p(t,T)$ is the value of a default-free ZCB with the same features, which is calculated using formulas (23)-(26). $\mathbf{Q}_t(\tau^H \leq T)$ is the probability of default under the risk-neutral measure.

This probability can be evaluated analytically (see Moraux, 2004):

$$\mathbf{Q}_t(\tau^H \leq T) = N\left[-\frac{\ln\frac{X_t}{H} + \mu\tau}{\sigma_X\sqrt{\tau}}\right] + \left(\frac{X_t}{H}\right)^{-2\mu/\sigma_X^2} N\left[-\frac{\ln\frac{X_t}{H} - \mu\tau}{\sigma_X\sqrt{\tau}}\right], \quad (33)$$

where $\mu = \alpha - \sigma_X^2 / 2$, $\tau = T - t$, $\tau^H = \inf\{t \geq 0 : X_t \leq H\}$, and $N(\cdot)$ is the cumulative distribution function of a standard Gaussian random variable.

In this model, credit spreads can be easily calculated

$$cs = -\frac{\ln[v(X,r;t,T)/p(t,T)]}{T-t} = -\frac{\ln[1 - w\mathbf{Q}_t(\tau^H \leq T)]}{T-t} \quad (34)$$

and do not depend on the level of interest rates.

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