Financing and Valuation of a Marginal Project by a Firm Facing Various Tax Rates

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Abstract

A multinational firm operating under various tax regimes can minimize the total after-tax cost of its debt by allocating it optimally between its projects. To value a marginal project in this context, we build a multi-period model for the selection of projects, assuming that the firm maintains a target debt ratio on a firm-wide basis. To define a project's adjusted present value, we successively adopt the Miles-Ezzell analysis and the Harris-Pringle analysis. We show that a project's net present value results from the addition of two sums of cash flows discounted at the firm's (marginal) after-tax WACC: the sum of the discounted expected after-tax operating cash flows and a sum of discounted differences in expected interest tax shields (multiplied by a simple adjustment factor in the Miles-Ezzell case). This valuation formula extends the field of application of the standard WACC method, since it can be applied to a project whose debt ratio differs from the firm's target debt ratio.

Keywords: interest tax shields, adjusted present value, WACC, debt allocation, debt ratio

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1 - Introduction

The issue addressed in this paper is the valuation of a marginal investment project by a multinational firm operating under various tax regimes. The firm is assumed to set a target debt-to-value ratio on the corporate scale, for all projects in the same risk class. As the fiscal treatment of interest payments is susceptible to vary significantly from one project to another, the firm will take advantage of the fact that certain projects have access to loans at a lower after-tax cost than is available elsewhere. However, standard methods, as discussed by Chambers et al. (1982), cannot be used to value a project of which the financing differs from the firm's target debt ratio.

This issue is especially important in the oil and gas production sector, where an international company has to contend with a large number of different tax regimes (see for example Johnston (2003)), as each country has its own petroleum tax code. For instance, in a Norwegian oil concession, the entire interest payment can be deducted from the taxable income for loans covering up to 80% of the capital investment. Since the marginal tax rate (ordinary income tax plus special petroleum tax) is 78%, high interest tax shields can be generated. On the other hand, in an Angolan production-sharing contract, no interest tax shields are generated, as interest payments are neither deducted from the taxable income nor recovered in the cost oil[1]. In other words, the after-tax cost of debt is equal to the before-tax cost for Angolan projects, whereas it is much lower for Norwegian projects. Let us consider a firm operating in both countries and satisfying a target debt ratio on the corporate scale. To minimize the total after-tax cost of its debt, this firm should finance Norwegian oil-field development projects with loans representing proportionally more than the fraction corresponding to its target debt ratio, whereas for Angolan projects the loans should be proportionally less.

To take this debt allocation process into account, Babusiaux and Pierru (2001) suggest adjusting the standard WACC method. Using a simple marginal analysis, they recommend adding to the operating cash flow a difference in after-tax interest payments (i.e., a difference in interest tax shields when the (before-tax) interest rate is the same for all projects, as is

[1] In a production-sharing contract, the cost oil is a portion of the production given to the firm in reimbursement of the costs incurred. It is equal to the sum of operating costs, depreciation and, in some countries, interest payments.

assumed here). Nevertheless, their demonstration, based on an analogy with subsidized loans, resorts to a formula which is not that recommended by the literature (see for instance Jennergren (2005)). Additionally, they do not study the consistency of their results with the adjusted present value method which adjusts for financing in one or more separate discounted cash flow terms.

In this paper, we develop a non-linear multi-period model for the selection of projects in the presence of various tax rates. The firm's value, equal to the sum of the adjusted present values of the projects selected, is maximized, under the constraint of satisfying a given target debt ratio at the corporate level. To define the adjusted present value, we successively adopt the Miles-Ezzell analysis (interest tax shields certain over one period) and the Harris-Pringle analysis (interest tax shields always uncertain). In each case, the Lagrange multipliers associated with the various constraints are determined at the optimal financing allocation. They enable the evaluation of a marginal project and are therefore useful for decision-making purposes. Our contributions are to solve the problem using the adjusted present value method (under various assumptions about the risk of interest tax shields), and to relate our results to the adjustment of the standard WACC method suggested by Babusiaux and Pierru.

2 – Assumptions and economy of the model

For the ease of discussion, we consider an international firm involved in the development and production of oil fields. The model, which covers a period of T years, maximizes the value of the firm at year 0. The firm has z investment opportunities (i.e., projects) that can be undertaken over this period. All these projects are assumed to be of the same operating risk and therefore have the same unlevered cost of equity ρ.

The firm has the possibility of choosing the (non-negative) size x_u of each project P_u, with:

$$x_u \geq 0 \quad u = 1, ..., z \tag{1}$$

This assumption is based on two observations:

- usually, a large oil field is developed by an association of several firms that share capital costs (in order to reduce their individual exposure to political risks, for instance); consequently, a given firm can consider a project as divisible;

- the decision to develop a large oil field may also include developing smaller surrounding oil fields and the sizing of a collecting pipe (with non-constant returns).

Causal or "physical" interdependencies between the cash flows of the various projects are not considered in the model. However, even if such interdependencies exist in real-life projects - for instance the simultaneous development of several oil fields in the same area may give rise to an increase in the corresponding investment costs -, their impact on the issue raised here, i.e., the debt allocation process, is likely to be negligible. To sum up, in the model, the projects constitute continuous ranges of investment independent from each other.

The expected after-tax operating cash flow[2] to be produced in year n ($n \in \left\{0,1,...,T\right\}$) by the project P_u ($u \in \left\{1,2,...,z\right\}$) is denoted $F_{u,n}\left(x_u\right)$ and is assumed to be differentiable with respect to x_u. Given these assumptions, to define bounded project sizes, simply consider the functions $F_{u,n}\left(x_u\right)$ so that, in the neighborhood of the bound, the investment cost grows very rapidly (or revenues fall sharply). Certain projects can be undertaken beginning in year 0, and others in later years. For a project P_q starting in year t, one should then take $F_{q,k}\left(x_q\right)=0$ for $k<t$. The expected terminal value of a project P_u in year T is given as a function of x_u and is denoted $V_{u,T}\left(x_u\right)$.

All the debt, assumed free of default risk, is contracted at the risk-free interest rate r. The loan expected to be allocated in year n to the project P_u is denoted $B_{u,n}$. As we only consider debt as a source of financing, we must have:

$$B_{u,n} \geq 0 \quad n=0,....,T-1, \quad u=1,....,z \tag{2}$$

As is specified later, the value of each project is calculated according to the adjusted present value, by imputing to the project all the expected after-tax operating cash flows and interest tax shields that are associated with it.

[2] Here, the term "after-tax operating cash flow" refers to the cash flow of the project before any financial claims are paid. For tax purposes, the taxable income used is defined as the earnings before interest and taxes, which means that the after-tax operating cash flow includes no interest tax shields.

The impact of debt on the value of the firm is therefore equal to the present value of the associated interest tax shields.

For each project P_u, the amount of interest tax shields generated in year n is taken to equal $\theta_u r B_{u,n-1}$. The interest payment $rB_{u,n-1}$ is thus assumed to be deductible from the project's taxable income which is subject to the tax rate[3] θ_u.

In real-life projects, the amount of debt for which interest payments are deductible from the taxable income cannot exceed a certain limit imposed by the local fiscal authorities. This maximum amount can be generally considered as a function of the size of the project. For instance, it is equal to 80% of the investment expenditure in a Norwegian[4] oil concession, as discussed in the introduction. In the model, this maximum loan amount, denoted $B^l_{u,n}\left(x_u\right)$, is assumed to be a continuous function differentiable with respect to x_u. The following constraints are included in the model:

$$B_{u,n} - B^l_{u,n}\left(x_u\right) \leq 0 \quad n = 0,....,T-1, \quad u = 1,....,z \tag{3}$$

Each constraint of type (3) means that a debt in excess of $B^l_{u,n}\left(x_u\right)$ generates no interest tax shields and, consequently, has no economic interest for the firm. Therefore, such a debt is not contracted through the optimal debt allocation process (even if this possibility were explicitly taken into account in the model).

Let $V_{u,n}$ denote the value of the project P_u at the end of year n (determined with the adjusted present value). At any given year, the value of the firm is considered as equal to the sum of the values of the entire set of projects undertaken. As the firm undertakes to satisfy a target debt ratio w on the corporate scale each year, we must have:

$$\sum_{u=1}^{z} B_{u,n} - w\sum_{u=1}^{z} V_{u,n} = 0 \quad n = 0,....,T-1 \tag{4}$$

[3] This tax rate depends on the fiscal rules specific to the project. If the interest payment is not deductible, as in the Angolan production-sharing contract mentioned in the introduction, then one should simply set out: $\theta_u = 0$.

[4] We voluntarily ignore here certain intricacies of the Norwegian petroleum tax system.

Concerning this assumption, Graham and Harvey (2001) report that about 80% of firms have some form of target debt-to-value ratio, and that the range around the target is tighter for larger firms.

Note that $V_{u,n}$ is not subject to a non-negativity constraint, since the value of a project can be negative during certain years. Such is the case for instance if the last cash flows of the project correspond to the cost for decommissioning and site rehabilitation of a depleted oil field.

In the model, each project is initially valued on a stand-alone basis, by attributing to it all the associated interest tax shields. The first-order optimality conditions – which take into account the equations of type (4) - will ensure the sharing of these tax shields between the various projects.

As the firm maintains a constant debt ratio, the future debt levels are not known. Therefore, the interest tax shields are not certain and cannot be discounted at the risk-free interest rate of the debt. To value a project, we successively adopt two distinct analyses:

The Miles-Ezzell analysis

According to Miles and Ezzell (1985), due to the expectation revision process, the adjusted present value $V_{u,n}$ of a given project P_u at the end of year n is :

$$V_{u,n} = \sum_{k=n+1}^{T} \left(\frac{F_{u,k}(x_u)}{(1+\rho)^{k-n}} + \frac{\theta_u r B_{u,k-1}}{(1+r)(1+\rho)^{k-n-1}} \right) + \frac{V_{u,T}(x_u)}{(1+\rho)^{T-n}}$$

$V_{u,n}$ and $V_{u,n+1}$ are therefore related by the following equation:

$$V_{u,n} = \frac{V_{u,n+1} + F_{u,n+1}(x_u)}{1+\rho} + \frac{\theta_u r B_{u,n}}{1+r} \quad u = 1,....,z \quad n = 0,....,T-2$$

$$V_{u,T-1} = \frac{V_{u,T}(x_u) + F_{u,T}(x_u)}{1+\rho} + \frac{\theta_u r B_{u,T-1}}{1+r} \quad u = 1,....,z$$

Or, equivalently:

$$(1+\rho)V_{u,n} - V_{u,n+1} - \left(\frac{1+\rho}{1+r}\right)\theta_u r B_{u,n} - F_{u,n+1}(x_u) = 0 \quad u = 1,....,z \quad n = 0,....,T-2 \qquad (5)$$

$$(1+\rho)V_{u,T-1} - V_{u,T}(x_u) - \left(\frac{1+\rho}{1+r}\right)\theta_u r B_{u,T-1} - F_{u,T}(x_u) = 0 \quad u = 1,....,z \qquad (6)$$

In the remainder of this article, $i_{m\text{-}e}$ denotes the corresponding adjusted discount rate that has to be used with the standard WACC method.

The Harris-Pringle analysis

Harris and Pringle (1985) have proposed that all interest tax shields be treated as risky and discounted at ρ. We therefore have:

$$V_{u,n} = \sum_{k=n+1}^{T} \frac{F_{u,k}\left(x_u\right) + \theta_u r B_{u,k-1}}{\left(1+\rho\right)^{k-n}} + \frac{V_{u,T}\left(x_u\right)}{\left(1+\rho\right)^{T-n}}$$

Hence:

$$\left(1+\rho\right)V_{u,n} - V_{u,n+1} - F_{u,n+1}\left(x_u\right) - \theta_u r B_{u,n} = 0 \quad u = 1,....,z \quad n = 0,....,T-2 \quad (7)$$

$$\left(1+\rho\right)V_{u,T-1} - V_{u,T}\left(x_u\right) - \theta_u r B_{u,T-1} - F_{u,T}\left(x_u\right) = 0 \quad u = 1,....,z \quad (8)$$

As shown by Taggart (1991), the corresponding adjusted discount rate (to use with the standard WACC method), denoted $i_{h\text{-}p}$ in this article, is a continuous-time version of $i_{m\text{-}e}$.

In short, the following notations, expressed in nominal dollars, have been adopted:

x_u	size of the project P_u
$F_{u,n}\left(x_u\right)$	expected after-tax operating cash flow produced in year n by project P_u
$V_{u,n}$	value of the project P_u at the end of year n
r	(risk free) interest rate of the loans
$B_{u,n}$	expected loan amount to be allocated to the project P_u at the end of year n
θ_u	tax rate at which $B_{u,n}$ generates interest tax shields
$B^l_{u,n}\left(x_u\right)$	maximum value that can be reached by $B_{u,n}$
ρ	cost of equity of an unlevered project
w	debt-to-value ratio set by the firm
T	horizon (in years) of the model
$i_{m\text{-}e}$	adjusted discount rate derived by Miles and Ezzell
$i_{h\text{-}p}$	adjusted discount rate derived by Harris and Pringle.

3 – Valuation of a marginal project consistent with the Miles-Ezzell analysis

3.1 Initial firm-value maximization problem

The firm-value maximization problem includes all constraints introduced in section 2. It is written as follows:

$$Max \sum_{u=1}^{Z}\left(V_{u,0}+F_{u,0}\left(x_u\right)\right)$$

$$s.t.\begin{cases}(u=1,....,z)\quad x_u\geq 0 & (1)\\ (u=1,....,z)\ (n=0,....,T-1)\quad B_{u,n}\geq 0 & (2)\\ (u=1,....,z)\ (n=0,....,T-1)\quad B_{u,n}-B^{l}_{u,n}\left(x_u\right)\leq 0 & (3)\\ (n=0,....,T-1)\quad \sum_{u=1}^{z}B_{u,n}-w\sum_{u=1}^{z}V_{u,n}=0 & (4)\\ (u=1,....,z)\ (n=0,....,T-2)\quad (1+\rho)V_{u,n}-V_{u,n+1}-\left(\dfrac{1+\rho}{1+r}\right)\theta_u rB_{u,n}-F_{u,n+1}\left(x_u\right)=0 & (5)\\ (u=1,....,z)\quad (1+\rho)V_{u,T-1}-V_{u,T}\left(x_u\right)-\left(\dfrac{1+\rho}{1+r}\right)\theta_u rB_{u,T-1}-F_{u,T}\left(x_u\right)=0 & (6)\end{cases}$$

This is a nonlinear problem, subject to both equality and inequality constraints. It includes three[5] types of variables: x_u, $B_{u,n}$ and $V_{u,n}$.

3.2 Dual relations satisfied by the Lagrange multipliers of the problem

We use Kuhn-Tucker conditions to determine the values of the Lagrange multipliers associated with the various constraints of the problem. We first derive the dual relations satisfied by these Lagrange Multipliers.

Let $\lambda_{u,n}$, μ_n, $\pi_{u,n}$ and $\pi_{u,T-1}$ be the Lagrange multipliers associated with constraints of type (3), (4), (5) and (6) respectively. To write the dual relations, we do not explicitly consider the Lagrange multipliers associated

[5] Even if, in practice, $V_{u,n}$ is an endogenous variable (and not a decision variable) of the problem.

with the non-negativity constraints of type (1) and (2). By applying the Kuhn-Tucker first-order optimality conditions, we obtain the following dual relations:

$$\begin{cases} (u=1,....,z)\ (n=0,....,T-1)\quad \lambda_{u,n} \geq 0 & (9) \\ (u=1,....,z)\quad 1-\pi_{u,0}(1+\rho)+w\mu_0 = 0 & (10) \\ (u=1,....,z)\ (n=0,....,T-1)\quad -\lambda_{u,n}+\left(\dfrac{1+\rho}{1+r}\right)\theta_u r\pi_{u,n}-\mu_n \leq 0 & (11) \\ (u=1,....,z)\ (n=1,....,T-1)\quad -(1+\rho)\pi_{u,n}+\pi_{u,n-1}+w\mu_n = 0 & (12) \\ (u=1,....,z)\quad \dfrac{dF_{u,0}(x_u)}{dx_u}+\sum_{n=1}^{T}\left(\pi_{u,n-1}\dfrac{dF_{u,n}(x_u)}{dx_u}+\lambda_{u,n-1}\dfrac{dB^l_{u,n-1}(x_u)}{dx_u}\right)+\pi_{u,T-1}\dfrac{dV_{u,T}(x_u)}{dx_u} \leq 0 & (13) \end{cases}$$

Equations of type (11) and type (13) are obtained by differentiating the Lagrangian of the problem with respect to variables $B_{u,n}$ and x_u respectively. They are inequalities as these variables are subject to non-negativity constraints.

Equations of type (10) and type (12) result from the differentiation of the Lagrangian of the problem with respect to variables $V_{u,0}$ and $V_{u,n}$ respectively.

Equations of type (9) indicate that $\lambda_{u,n}$ must be non negative, as this Lagrange multiplier is associated with an inequality constraint.

3.3 Solution to the dual relations

In year 0 (as in other years), the firm first allocates a loan to the project which generates the highest interest tax shield. Type (3) constraint corresponding to this project is then binding. The firm then considers other possibilities for debt allocation in decreasing tax shield (i.e., decreasing θ_u) order. We assume there is one project P_v for which $0 < B_{v,0} < B^l_{v,0}(x_v)$. In other words, the last dollar of debt allocated in the firm is attributed to the project P_v. On an after-tax basis, the firm's marginal cost of debt in year 0 is therefore $(1-\theta_v)r$. Constraints (2) and (3) corresponding to the project P_v are therefore not binding. Consequently, we have $\lambda_{v,0} = 0$ and equation (11) is the following equality:

$$\left(\frac{1+\rho}{1+r}\right)\theta_v r \pi_{v,0} - \mu_0 = 0$$

For this project, equation (10) is written:

$$1 - \pi_{v,0}\left(1+\rho\right) + w\mu_0 = 0$$

Combining these two equalities gives us:

$$\pi_{v,0} = \frac{1}{1+\rho - w\theta_v r\left(\frac{1+\rho}{1+r}\right)} \text{ and } \mu_0 = \frac{\theta_v r}{1+\left(1-w\theta_v\right)r}$$

According to equations of type (10), we have:

$$\pi_{u,0} = \pi_{v,0} \qquad u = 1,\ldots,z$$

Let us consider a project P_u to which a loan is allocated in year 0 (i.e., we have $\theta_u \geq \theta_v$). This project has a non-binding type (2) constraint: $B_{u,0} > 0$. Equation (11) is therefore an equality which gives us:

$$\lambda_{u,0} = \frac{\left(\theta_u - \theta_v\right)r}{1+\left(1-w\theta_v\right)r}$$

We now generalize these results to the following years. To simplify notations, we assume that every year the same project P_v serves to define "the firm's marginal tax rate" θ_v (i.e., the tax rate at which the last dollar of debt allocated in the firm generates a tax shield). For any year *n*, we then have:

$$\left(\frac{1+\rho}{1+r}\right)\theta_v r \pi_{v,n} - \mu_n = 0$$

By combining this equation with equation (12) for project P_v, we obtain:

$$\frac{\pi_{v,n}}{\pi_{v,n-1}} = \frac{1}{1+\rho - w\theta_v r\left(\frac{1+\rho}{1+r}\right)}$$

As $\pi_{v,0}$ has already been determined, we immediately have:

$$\pi_{v,n} = \frac{1}{\left(1+\rho - w\theta_v r\left(\frac{1+\rho}{1+r}\right)\right)^{n+1}}$$

Consider any two projects P$_q$ and P$_w$. Equation (12) proves that if $\pi_{q,n-1} = \pi_{w,n-1}$ then $\pi_{q,n} = \pi_{w,n}$. Since this assumption was verified for year 0, there is indeed equality of these Lagrange multipliers each year. For any project P$_u$ we therefore have:

$$\pi_{u,n} = \frac{1}{\left(1+\rho - w\theta_v r\left(\frac{1+\rho}{1+r}\right)\right)^{n+1}} \quad n = 0,....,T-1 \tag{14}$$

The term on the right-hand side of equation (14) is the adjusted discount rate formula derived by Miles and Ezzell (1980) (see also Taggart (1991), Brealey and Myers (2003)), which is here calculated using the firm's target debt ratio and the firm's marginal tax rate. Therefore, by setting:

$$i_{m-e} = \rho - w\theta_v r\left(\frac{1+\rho}{1+r}\right)$$

We have:
$$\pi_{u,n} = \frac{1}{\left(1+i_{m-e}\right)^{n+1}}$$

Projects (other than P$_v$) to which a loan is allocated in year n have their constraint (3) binding. Equation (11) is then an equality, which allows us to write:

$$\lambda_{u,n} = \left(\frac{1+\rho}{1+r}\right)\frac{\left(\theta_u-\theta_v\right)r}{\left(1+\rho - w\theta_v r\left(\frac{1+\rho}{1+r}\right)\right)^{n+1}} = \left(\frac{1+\rho}{1+r}\right)\frac{\left(\theta_u-\theta_v\right)r}{\left(1+i_{m-e}\right)^{n+1}} \tag{15}$$

We can now replace the Lagrange multipliers by their respective values in equation (13):

$$\frac{dF_{u,0}\left(x_u\right)}{dx_u} + \sum_{n=1}^{T}\frac{\frac{dF_{u,n}\left(x_u\right)}{dx_u} + \left(\frac{1+\rho}{1+r}\right)\left(\theta_u-\theta_v\right)r\,\frac{dB^l_{u,n-1}\left(x_u\right)}{dx_u}}{\left(1+i_{\text{m-e}}\right)^n} + \frac{\frac{dV_{u,T}\left(x_u\right)}{dx_u}}{\left(1+i_{\text{m-e}}\right)^T} \le 0 \tag{16}$$

3.4 Economic interpretation

The valuation of a marginal project is made possible by equation (16) which takes the interactions of investment and financing decisions into account when a project's size increases. This equation was derived from equation (13) by replacing the Lagrange multipliers with their values. Let us clarify the economic meaning of these Lagrange multipliers and interpret equation (16).

First, the Miles and Ezzell's formula of the adjusted discount rate to use with the standard WACC method is here obtained, in an original way, as a Lagrange multiplier of the firm-value maximization problem ($\pi_{u,n}$ in equation (14)). As a result of the optimal debt allocation process, this formula is computed using data relating to the firm's marginal after-tax cost of debt. Specifically, the tax rate used is the firm's marginal tax rate (at which the last dollar of debt allocated in the firm generates a tax shield). This result is consistent with the suggestion made by Babusiaux and Pierru (2001) to compute the firm's after-tax weighted average cost of capital with this marginal tax rate.

In the problem, $\pi_{u,n}$ is associated with a constraint of type (5). An increase in the right-hand-side coefficient of this constraint implicitly corresponds to an increase in the after-tax operating cash flow. Consequently, in equation (16), the increase in the after-tax operating cash flow caused by a change in the project size varies the firm's value by an amount equal to this increase discounted at the adjusted discount rate.

The Lagrange multiplier $\lambda_{u,n}$ represents the increase in value of the firm if it has the option of allocating one additional dollar -generating an interest tax shield at the tax rate θ_u - to project P_u in year *n*. In this case, as the firm's debt ratio is maintained equal to *w*, one dollar of debt is implicitly displaced from project P_v (that serves to define the firm's marginal tax rate) to project P_u. According to equation (15), the increase in the firm's value is then equal to the corresponding difference in interest tax shields multiplied by the factor $\frac{1+\rho}{1+r}$ and discounted at the adjusted discount rate. In equation (16), this increase is multiplied by the change in the amount of tax-deductible debt caused by a change in the size of the project.

The term on the left-hand side of equation (16) can be either negative or equal to zero. The interpretation is straightforward: the net present value of the last dollar invested in project P_u is negative or equal to zero. There is indeed optimum project sizing. An unrealized project, i.e., a project P_u for which $x_u = 0$, has a negative or zero marginal net present value. The marginal net present value of a realized project (i.e., a project P_u for which $x_u > 0$) is equal to zero.

To sum up, the net present value of a marginal project is obtained by considering, each year, the sum of the after-tax operating cash flow and an interest-tax-shield adjustment. It has to be noted that the factor $\frac{1+\rho}{1+r}$ is not included in the adjustment recommended by Babusiaux and Pierru (2001). Therefore, these authors slightly underestimate the correct adjustment if we adhere to the Miles-Ezzell analysis.

Theoretically, the valuation formula provided by equation (16) allows the firm to decentralize the financing decision. To maximize the value of the project or that of the firm clearly leads to the same decision: a loan is allocated to project P_u if, and only if, $\theta_u \geq \theta_v$ (otherwise the interest-tax-shield adjustment would be a negative cash flow).

4 – Valuation of a marginal project consistent with the Harris-Pringle analysis

The initial problem is very similar to that presented in the previous section. Only equations of type (5) and (6) have to be replaced by equations of type (7) and (8). The dual relations can be solved in the same way. We obtain:

$$\pi_{u,n} = \frac{1}{\left(1+\rho - w\theta_v r\right)^{n+1}} \qquad u = 1,....,z \quad n = 0,....,T-1$$

In this equation, we recognize the expression of the adjusted discount rate derived by Harris and Pringle (1985) and commented by Taggart (1991), calculated using the firm's target debt ratio and the firm's marginal tax rate . As previously, this adjusted discount rate can be considered as the firm's after-tax weighted average cost of capital computed on a marginal basis.

By setting $i_{h-p} = \rho - w\theta_v r$, we therefore have for any project P_u:

$$\pi_{u,n} = \frac{1}{\left(1+i_{h-p}\right)^{n+1}} \qquad n = 0,....,T-1$$

We also have:

$$\lambda_{u,n} = \frac{(\theta_u - \theta_v) r}{(1 + i_{h-p})^{n+1}} \qquad n = 0, ..., T - 1$$

Equation (16) in the previous section is now written as follows:

$$\frac{dF_{u,0}(x_u)}{dx_u} + \sum_{n=1}^{T} \frac{\frac{dF_{u,n}(x_u)}{dx_u} + (\theta_u - \theta_v) r \frac{dB^l_{u,n-1}(x_u)}{dx_u}}{(1 + i_{\text{h-p}})^n} + \frac{\frac{dV_{u,T}(x_u)}{dx_u}}{(1 + i_{\text{h-p}})^T} \leq 0 \qquad (17)$$

Equation (17) corresponds to the generalized ATWACC method suggested by Babusiaux and Pierru (2001): a marginal project's value is equal to the sum of the after-tax operating cash flows and differences in interest tax shields discounted at the firm's marginal after-tax weighted average cost of capital.

5 – Conclusion

When corporate financing and investment decisions interact, as is clearly the case here, the financial literature recommends using the adjusted present value method. From our model, based on this method, the value of a marginal project results from the addition of two sums of cash flows discounted at the firm's (marginal) after-tax WACC:
- the sum of the expected after-tax operating cash flows,
- a sum of differences in interest tax shields.

If we adhere to the Miles-Ezzell analysis, the latter term has to be multiplied by $\frac{1+\rho}{1+r}$.

This valuation formula extends the field of application of the standard WACC method since it can be applied to a project of which the debt ratio differs from the firm's target debt ratio. Note that the firm’s use of a single discount rate implies that the cost of equity is the same for all projects, regardless of their debt ratios, which may appear to contradict some traditional results of finance theory. In fact, this is induced by the constraint of compliance with the target debt ratio at firm's level. This constraint "connects" the projects and results in no differentiation in their cost of equity.

The adjustment of the standard WACC method suggested by Babusiaux and Pierru (2001), called "generalized ATWACC method", is therefore consistent with our findings. At worse, these authors may slightly underestimate the present value of the adjustment, the magnitude of this underestimation depending on the unlevered cost of equity.

Our results exactly match the generalized ATWACC method if we follow the Harris-Pringle analysis. They can be easily extended to the case of a risky debt by using the formulas derived by Ruback (2002), who discounts at the unlevered cost of equity the interest tax shields generated by a debt with a non-zero beta.

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