Interest Rate Risk in a Negative Yielding World

Joel R. Barber[1]

Krishnan Dandapani[2]

Abstract

Duration is widely used in the financial services industry to measure and manage interest rate risk. Both the development and the empirical testing of duration occurred in an environment with positive interest rates. This raises the question of how well standard duration analysis performs in a negative yielding world. We examine the mathematical properties of duration and convexity in a negatively yielding world. In particular, we identify the properties that are unique to a negative-yield environment and properties that are the same regardless of the sign of the yield. For the most part, if a bond has a positive or zero coupon rate, the mathematical properties of duration and convexity are the same. Because of the convexity property, the dollar sensitivity of the bond price is much higher at negative yields. If a bond has a negative coupon rate, the properties of duration and convexity are quite different.

Key Words: duration, convexity, yield curves
JEL Classification: G11

[1] Florida International University, E-Mail: barberj@fiu.edu, Telephone: 305-348-2680(USA), corresponding author
[2] Florida International University, E-Mail: dandapan@fiu.edu, Telephone: 305-348-2680(USA)

1 – Introduction

Currently, many countries are experimenting with negative interest rates. The value of negative-yielding bonds increased to $13.4 trillion in August 2016 (Wigglesworth, 2016). The volume of negative-yielding bonds could increase as the European Central Bank and the Bank of Japan plan additional bond purchases to revive their economies. Further, if the world economy is entering into an age of secular stagnation (Summers, 2015 and 2016), low and negative interest rates would be a long-term phenomenon that investors need to get a handle on.

Duration is widely used in the financial services industry to measure and manage interest rate risk. Both the development of duration[3] and the empirical testing of duration occurred in an environment with positive interest rates. This raises the question of how well standard duration analysis performs in a negative yielding world. We examine the mathematical properties of duration and convexity in a negatively yielding world.

The next section defines the terms and notation used in this paper. Section 3 describes the comparative static properties of the bond price function with respect to time changes and yield changes for bonds with positive or zero coupon rates. Section 4 expands this analysis by examining the accuracy of the commonly used Taylor Series approximation of bond price sensitivity based on duration and convexity. Section 5 examines duration and convexity for bonds with negative coupon rates. Section 6 concludes the paper.

2 - Notation and definitions

For simplicity, we assume that the cash flows are paid annually. This assumption is not required to obtain the results in this paper.

1. The continuously compounded yield to maturity is denoted by y.
2. The yield compounded once per year is determined by:

$$Y = e^{y} - 1$$

[3] See, Bierwag, Kaufman,Toevs (1983); Bierwag and Khang (1979); Bierwag (1977) Fisher and Weil (1971); Macaulay (1938); Redington (1952).

3. The price of a fixed income security is related to the yield as follows:

$$P = \sum_{j=1}^{T} C_j e^{-yj} \tag{1}$$

where C_1, C_2, …, C_T are the cash flows promised in years 1, 2, ... , T.

4. If R is the annual coupon rate for a bond with a par value of \$1 and time to maturity T, then based on equation (1) the closed-form expression for the price is:

$$P = \frac{R}{Y}(1 - e^{-yT}) + e^{-yT} \tag{2}$$

5. The duration[4] is defined as

$$D = -\frac{1}{P}\frac{dP}{dy} \tag{3}$$

evaluated at the currently observed yield to maturity.

6. The convexity is defined as

$$X = \frac{1}{P}\frac{d^2P}{dy^2} \tag{4}$$

evaluated at the currently observed yield to maturity.

3 - Positive or zero coupon rate bonds with negative yields

This section extends the study of interest rate risk to negative yielding bonds with positive or zero coupon rates. In particular, we identify the properties that are unique to a negative-yield environment and properties that are the same regardless of the sign of the yield.

Complete analysis of bond risks includes the passage of time as well as the effect of changes in bond yield.[5] Indeed, the comparative static analysis used in duration analysis breaks the price change into a component due to the

---

[4] Under continuous compounding the standard or Macaulay (1938) duration, Fisher-Weil (1971) duration, and modified duration (Fabozzi, 2007, 2008) are all the same.

[5] See Poitras, 2007; Barber and Copper (1997); Christensen and Sorensen, 1994; Reitano (1992); Chance, D. and J. Jordan (1996).

passage of time and a component due to changing interest rates. In particular, the total differential of the bond price can be expressed as:

$$dP = \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial y} dy$$

The first term measures the effect of the change in time assuming the yield does not change and the second term measures the effect of a change in the yield assuming time does not change.

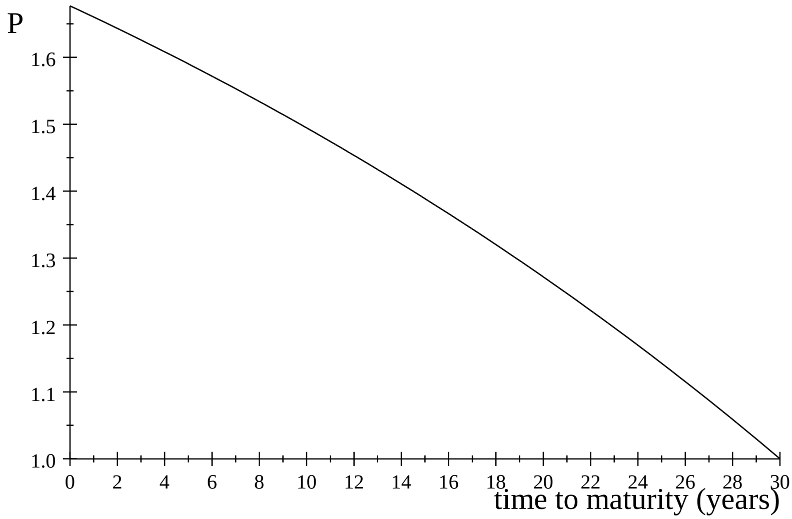
Property 1. Bonds with positive or zero coupon rates sell at a premium at negative yields.

In a negative yield environment, only bonds with negative coupon rates sell at a discount or par. All other bonds, including zero coupon bonds sell at a premium.

Property 2. The price path of premium bonds is a convex function of time to maturity at negative yields.

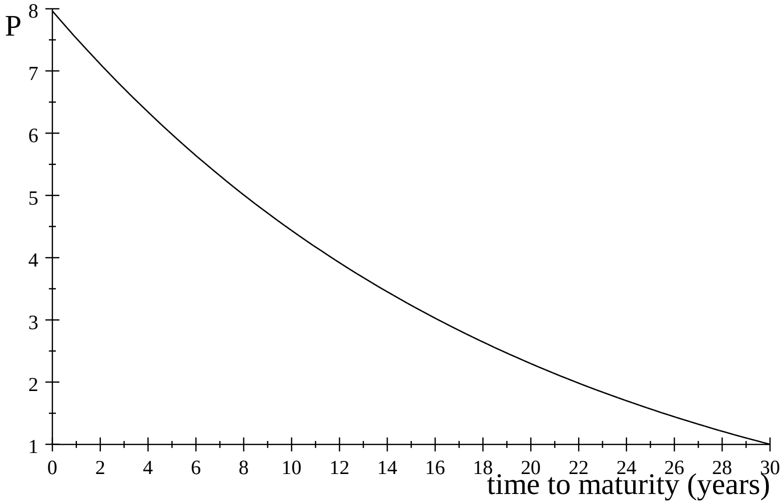
The effect of the passage of time is different for negative yielding bonds. For positive yields, it is well known that the price of a premium bond is a concave function of time to maturity. Figure 1 shows the price path (amortization pattern) of a 5%, 30 year coupon bond at constant yield of 2%.

Figure 1: Price Path for a 5%, 30-Year Bond at a Constant Yield of 2% per year



However, at negative yields the price is a convex function of the time to maturity. Figure 2 shows the price path for the same bond at a yield of −5%.

Figure 2: Price Path for a 5%, 30-Year Bond at a Constant Yield of -5%



We can easily prove this result. Based on equation (2), if the yield y is constant, the price of the bond a time t is given by:

$$P(t, y) = \frac{R}{Y}\left(1 - e^{-y(T-t)}\right) + e^{-y(T-t)}$$

Taking the partial derivative of P with respect to t, results in:

$$\frac{\partial P}{\partial t} = -\frac{y}{Y} e^{-y(T-t)}(R - Y) < 0$$

The inequality holds because the bond is selling at a premium ($c \geq 0 > y$) and the sign of y/Y is positive, as both the sign of the numerator and denominator are negative. The second derivative is given by:

$$\frac{\partial^2 P}{\partial t^2} = -y\left(\frac{y}{Y}\right) e^{-y(T-t)}(R - Y)$$

If a bond is selling at a premium, then the sign of the second derivative depends on the sign of the yield. For positive interest rates the sign is negative, implying that the function is concave. However, for negative rates the sign is positive, implying that the function is a convex function of time.

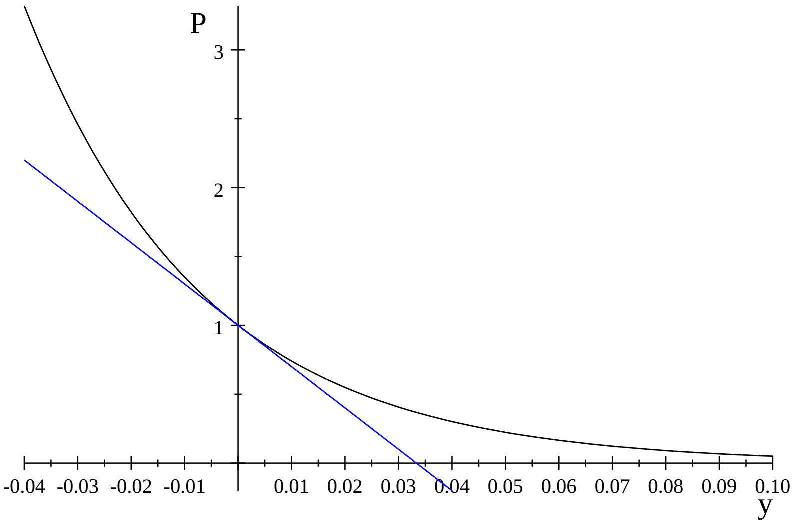
Given that the bond price, duration, and convexity are linear in terms of the vector of cash flows, it is reasonable to first analyze zero coupon bonds, and then treat coupon bonds as portfolios (or linear combinations) of zero-coupon bonds.

Property 3. Duration and convexity of zero coupon bonds are positive and equal to the time to maturity and time to maturity squared regardless of the sign of the yield.

Based on equations (3) and (4), it is easy to show that the duration of a zero coupon bond equals to the time to maturity and the convexity equals time to maturity squared for both positive and negative yields.

Property 3 means that the duration of a zero-coupon bond is a linear function of the time to maturity. Figure 3 shows the price-yield curve of a zero coupon bond with a maturity of 30 years. Clearly, the bond sells for par at a yield of 0%. Obviously, the curve is a decreasing and convex function of yield, because the zero coupon bond price is an exponential function of the yield. Because of this convexity, the dollar sensitivity (dollar duration) is greater at low interest rates. This is especially true for negative rates. Nevertheless, the relative (percentage) sensitivity is the same for all yields, because the duration is constant for zero coupon bonds.

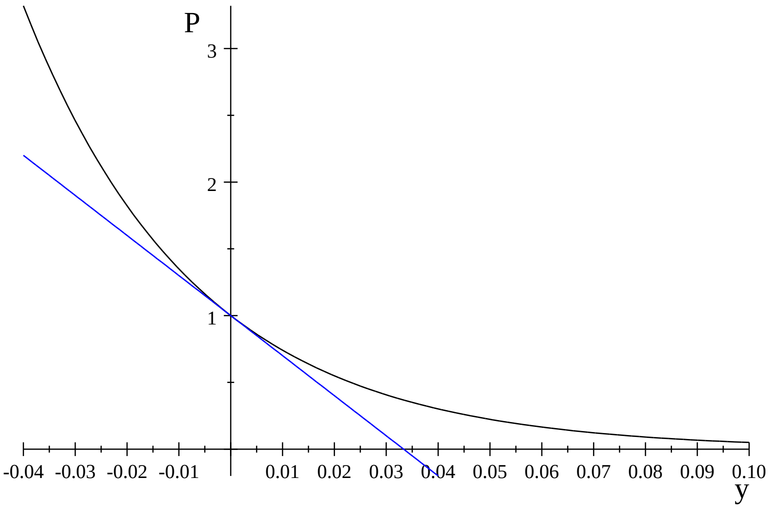
Figure 3: Price-Yield Curve of a Zero Coupon bond with maturity of 30 Years.



Property 4. For coupon bonds both the relative and absolute sensitivity of the bond price is higher at negative yields.

Since the coupon rate is positive, even though the yield is negative, the price of the bond is the sum of positive convex functions, and therefore the price is a convex function of the yield. Figure 4 shows the price-yield curve for a 30-year, 4% coupon bond. Again, the dollar sensitivity is much higher at negative rates.

Figure 4: Price-Yield Curve for a 30-year, 4% Coupon Bond



However, since the duration is also a function of the yield, the relative sensitivity of the coupon bond also changes with the yield. Both the absolute and relative sensitivity of the coupon bond is higher at a negative yield. For a coupon bond, we can show that

$$\frac{\partial D}{\partial y} < 0$$

Equation (1) states:

$$P(y) = \sum_{j=1}^{T} C_j e^{-yj}$$

Then

$$\frac{P'}{P} = -\sum_{j=1}^{T} \frac{C_j e^{-yj}}{P} j = -\sum_{j=1}^{T} w_j\, j = E[j]$$

and

$$\frac{P''}{P} = \sum_{j=1}^{T} \frac{C_j e^{-yj}}{P} j^2 = \sum_{j=1}^{T} w_j\, j^2 = E[j^2]$$

where the weights $w_j \geq 0$ (given the coupon rate is not negative) and sum to one. The expression $E[\ ]$ is the expectation treating the weights as probabilities. From the chain rule, we know

$$D' = -\frac{P''}{P'} + \left(\frac{P'}{P}\right)^2$$

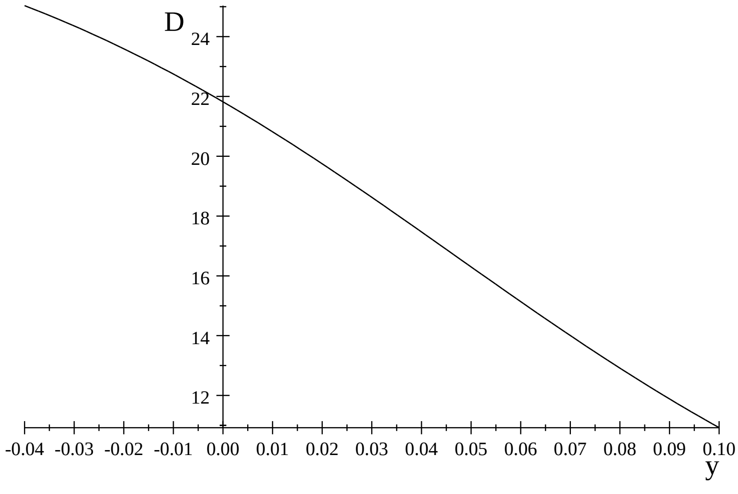
or

$$D' = -(E[j^2] - E^2[j]) \leq 0$$

The inequality, based on a well-known property of expecation, holds because the weights are non-negative and sum to one. For coupon bonds the inequality is strict. For zero coupon bonds the result is an equality because the only weight is equal to one. If the coupon rate is negative, the result does not hold.

To illustrate this, we plot the duration of a 4%, 30-year coupon bond versus the yield to maturity in Figure 5.

Figure 5: Duration of a 4%, 30-year Coupon Bond vs the Yield to Maturity



4 - Taylor series approximation to the price-yield curve

The sensitivity of a bond price to a change in yield is commonly approximated with a Taylor Series of the price as a function of the yield.[6] Suppose the yield changes from the currently observed value of y_0 to a future unknown value y. A first-order Taylor polynomial of the price P expanded about current yield to maturity relates the bond price in response to a change in yield as follows:

$$P(y) \approx P(y_0) - P(y_0)D(y - y_0) \quad (5)$$

For small yield changes, the duration D can be thought of as a measure of interest rate risk. A second-order Taylor polynomial approximation takes into account both the duration and the convexity X:

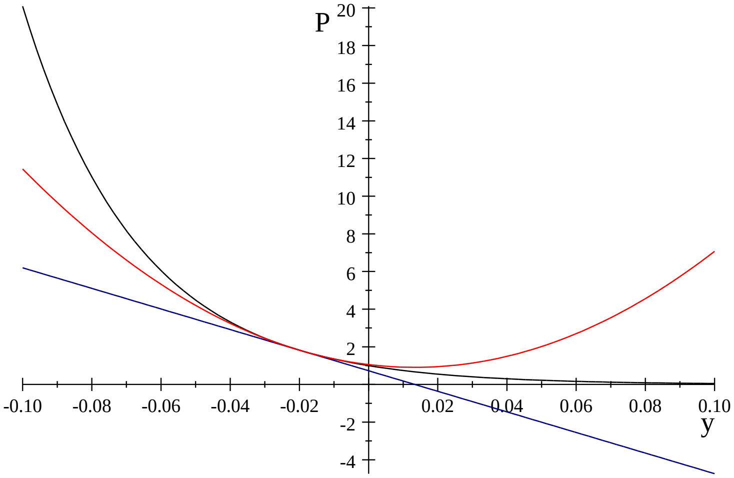
$$P(y) \approx P(y_0) - P(y_0)D(y - y_0) + \frac{1}{2}P(y_0)X(y - y_0)^2$$

The accuracy of the approximation depends on the size of the yield change. This is especially a concern for the duration approximation (5) because the convexity is much higher at negative yields. Figure 6 shows the first- and second-order Taylor series polynomial approximation of the price-yield curve for 30-year zero coupon bond expanded about a yield of -2% in Panel A, a yield of 0% in Panel B, and a yield of 2% in Panel C. The top curve on the left side is the true price-yield curve. We can see that both approximations are closer to the true value at a positive yield of 2%.

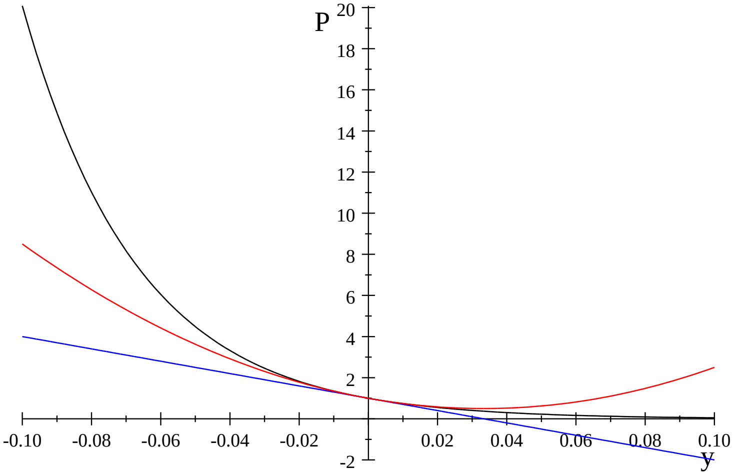
[6] See Bierwag, Kaufman, and Latta (1988); Winkelmann (1989); Barber (1995); Livingston and Zhou (2005); Fabozzi (2008).

Figure 6: First and Second-order Taylor Series Polynomial Approximation and True Price-Yield Curve

Panel A: Expansion about yield of -2.0%



Panel B: Expansion about yield of 0%



Panel C: Expansion about yield of 2.0%

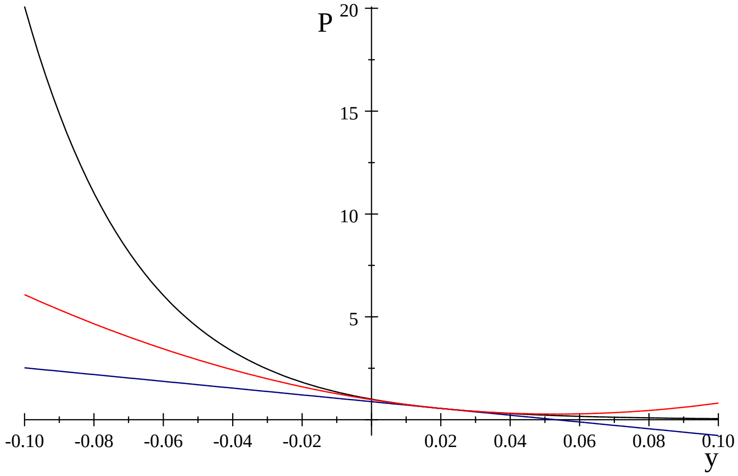


Table 1 computes the root-mean approximation square error[7] (RMSE) over a range of yield changes for 1st-order and 2nd-order Taylor polynomial approximations of zero-coupon bond price changes. The results are presented for maturities of 5, 10, 15, and 30 years. As expected, the RMSE of the approximation centered at a negative yield is greater. This is especially true for bonds with long maturities and large yield ranges. For example, for a yield range of ±100 basis points the difference between the 1st-order RMSE centered at a yield of −2.0% versus 2.0% is \$1.12 (per \$10,000 par) for a maturity of five years. For a 10-year maturity this difference in RMSE increases to \$8.92, and for maturities of 15 and 30 years the differences are \$30.70 and \$258.54, respectively.

[7] The root-mean square error centered about yield y_0 for a range of changes $\pm\Delta$ is determined by:

$$\sqrt{\frac{1}{2\Delta}\int_{y_0-\Delta}^{y_0+\Delta}(P(y)-P_0(y))^2dy}$$

where P_0 is the Taylor series approximation centered about y_0.

Table 1: Root-Mean Square Approximation Error (in $s) over a Range of Yields for 1st-order and 2nd-order Taylor Polynomial of Zero-Coupon Bond Price

Change is centered about −2.0%, 0.0%, and +2.0%. The bond has a par value of $10,000.

Panel A: Maturity of 5 years						
	Centered at −2.0%		Centered at 0.0%		Centered at +2.0%	
Range (bps)	1st-order	2nd-order	1st-order	2nd-order	1st-order	2nd-order
±25	0.39	0.00	0.35	0.00	0.32	0.00
±50	1. 54	0.01	1.40	0.00	1.26	0.01
±100	6.18	0.08	5.59	0.08	5.06	0.07
±200	24.74	0.70	22.38	0.63	20.25	0.57
Panel B: Maturity of 10 years						
	Centered at −2.0%		Centered at 0.0%		Centered at +2.0%	
Range (bps)	1st-order	2nd-order	1st-order	2nd-order	1st-order	2nd-order
±25	1.71	0.01	1.40	0.01	1.14	0.01
±50	6.83	0.10	5.59	0.09	4.58	0.06
±100	27.24	0.77	22.38	0.63	18.33	0.52
±200	109.68	6.17	89.80	5.05	73.52	4.14
Panel C: Maturity of 15 years						
	Centered at −2.0%		Centered at 0.0%		Centered at +2.0%	
Range (bps)	1st-order	2nd-order	1st-order	2nd-order	1st-order	2nd-order
±25	4.25	0.04	3.14	0.03	2.33	0.02
±50	17.00	0.36	12.59	0.27	9.32	0.20
±100	68.06	2.87	50.42	2.13	37.36	1.58
±200	274.08	23.09	203.05	17.10	150.42	12.67

Panel D: Maturity of 30 years						
	Centered at −2.0%		Centered at 0.0%		Centered at +2.0%	
Range (bps)	1st-order	2nd-order	1st-order	2nd-order	1st-order	2nd-order
±25	22.93	0.48	12.58	0.27	6.91	0.15
±50	91.88	3.88	50.42	2.13	27.67	1.17
±100	369.97	31.17	203.05	17.10	111.43	9.39
±200	1519.5	253.60	833.91	139.18	457.66	76.38

5 - Negative coupon rates

As far as we know, negative-coupon rate bonds have never been issued. Given that not too long ago, the same thing also could have been said about negative yielding bonds, it might be worthwhile to briefly consider bonds with negative coupon rates.

A negative coupon rate may be unrealistic because the holder of the bond would have to pay the issuer for the privilege of owning the bond. However, given that some floating rate loans currently require the lender to pay the borrower, we should not dismiss negative coupon rate bonds out of hand. Recently, banks in Denmark have been paying interest on adjustable-rate mortgages by deducting the amount of interest from the principal the customer owes. (Duxbury, 2016)**.** While not widespread, this practice is also occurring in Sweden, Portugal and Spain.

Another, more serious objection can be made to a negative coupon rate bond. In particular, it is possible that if yields increase the bond could sell for a negative price. This means in the secondary market, the seller of a bond would have to pay the buyer. In other words, a negative coupon rate bond is not a limited liability asset.[8] This reason alone is perhaps sufficient to argue that negative rate bonds will never be issued.

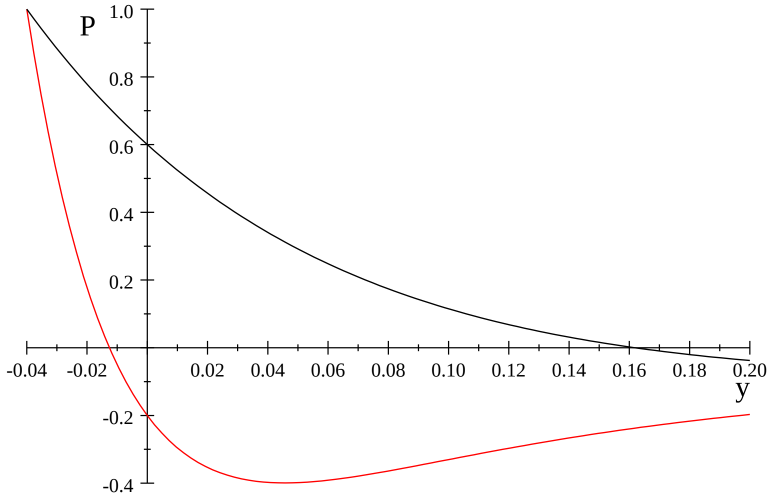
A bond with a negative coupon rate could sell for par value (or even a discounted value) at a negative yield, whereas a non-negative coupon rate bond always sells at a premium. It turns out, that negative coupon rate bonds have

[8] This is not true of floating rate (adjustable rate) bonds and mortgages.

mathematical properties that are quite different than positive or zero coupon bonds at both positive and negative yields. If all the cash flows are all positive, the bond price is a convex function of the yield, because the sum of positive convex functions is convex. However, if some of the cash flows are negative, this is no longer true. Given that most of the cash flow comes from the redemption of the par value at maturity, under reasonable conditions the price-yield curve would be convex over a wide range of yields.

Figure 7 shows the price-yield curve for −4% coupon bond with a 30-year (lower curve) and 10-year (upper curve) maturity. The negative convexity is more likely to show up for long maturity bonds with large magnitude negative coupon rates, and when interest rates are high. Figure 7 demonstrates that the price of a bond with a negative coupon rate could become negative if yields rise after the bond is issued. This figure also demonstrates the possibility of negative duration for a long-maturity bond with a negative coupon rate.

Figure 7: Price-Yield Curve for −4% Coupon Bond with a 30-Year (lower curve) and 10-Year (upper curve) Maturity



We can safely conclude that the negative duration and convexity of negative coupon rate bonds is more of a curiosity than a phenomenon that would be actually encountered.

6 – Conclusion

At negative yields, bonds with positive or zero coupon rates necessarily sell for a premium. Unlike bonds priced at positive yields, the price path of a premium bond is a convex function of time. The mathematical properties of duration and convexity are the same at negative yields for bonds with positive or zero coupon rates, but not the same for bonds with negative coupon rates, where both the duration and convexity could be negative. Because of the convexity property of bonds with positive or zero coupon rates, the dollar sensitivity of the bond price is much higher at negative yields. However, for zero coupon bonds the relative sensitivity (measured) by the duration is the same. For coupon bonds, the relative sensitivity is higher at negative rates. The reason for this is the duration of a coupon bond is a decreasing function of the yield. These results suggest that immunization strategies involving negative-yielding bonds would have to be rebalanced more frequently. The accuracy of the Taylor series approximation of the price-yield curve, based on duration alone or both duration and convexity, is some what lower at negative rates. In other words, the duration is a less accurate measure of interest rate risk at negative yields.

References

Barber, J. R. and M. Copper, 1997. Is bond convexity a free lunch?, Journal of Portfolio Management, 24: 113-119.

Barber, J. R.,1995. A note on approximating bond price sensitivity with duration and convexity, Journal of Fixed Income 4, no. 4, 95.

Bierwag, G. O., Kaufman, G. G., & Latta, C. M., 1988. Duration models: A taxonomy. Journal of Portfolio Management, 15(1), 50. Financial and Quantitative Analysis 13, no. 4: 671–81.

Bierwag, G. O., G. G. Kaufman, and A. Toevs, 1983. Bond portfolio immunization and stochastic process risk, Journal of Bank Research 13, no. 4, 282-291.

Bierwag, G. O. and C. Khang, 1979. An immunization strategy is a minimax strategy, Journal of Finance 34, 389-414.

Bierwag, G. O.,1977. Immunization, duration, and the term structure of interest rates, Journal of Financial and Quantitative Analysis 12, 725-742.

Chance, D. and J. Jordan, 1996. Duration, convexity and time as components of bond returns, Journal of Fixed Income, 6 (September): 88-96.

Christensen, P. O and B. G. Sorensen, 1994. Duration, convexity, and time value. Journal of Portfolio Management, 20(2), 51.

Duxbury C. and D. Gauthier-Villars Wall Street Journal, 2016 (April 15), Home borrowers collect interest, C-1.

Fabozzi, F. J.,2008. Bond markets, analysis and strategies, seventh edition, Prentice Hall. 565-593.

Fabozzi, F. J., 2007. Portfolio Immunization and cash flow matching. Chapter 19 in Fixed Income Analysis. 2nd Ed. Hoboken, NJ: John Wiley & Sons.

Fisher, L. and R. L. Weil, 1971. Coping with the risk of interest-rate fluctuations: returns to bondholders from naive and optimal strategies, Journal of Business 44, 408-431.

Livingston, M. and Zhou, L., 2005. Exponential duration: a more accurate estimation of interest rate risk, Journal of Financial Research, 28, 343–61.

Macaulay, F.,1938. Some theoretical problems suggested by movement of interest rates, bonds, yields, and stock prices in the United States since 1856, Columbia University Press, New York.

Poitras, G., 2007. Immunization bounds, time value and non-parallel yield curve Shifts, Insurance and Risk Management, 75(3), 323-356.

Redington, F. M.,1952. Review of the principle of life-office valuations. Journal of the Institute of Actuaries 78, 286-340.

Reitano, R. R.,1992. Nonparallel yield curve shifts and convexity, Transactions of the Society of Acturaries, Vol. XLIV, 479-507.

Summers, L. H., 2016. The age of secular stagnation. Foreign Affairs, 95(2), 2-9.

Summers, L. H., 2015. Demand side secular stagnation. The American Economic Review, 105(5), 60-65.

Wigglesworth, R. and E. Platt, 2016. Value of negative-yielding bonds hits $13.4tn, Financial Times.

Winkelmann, K., 1989. Uses and abuses of duration and convexity. Financial Analysts Journal, 45(5), 72.